

# SOFIC MEAN DIMENSION

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**ABSTRACT.** We introduce mean dimensions for continuous actions of countable sofic groups on compact metrizable spaces. These generalize the Gromov-Lindenstrauss-Weiss mean dimensions for actions of countable amenable groups, and are useful for distinguishing continuous actions of countable sofic groups with infinite entropy.

## 1. INTRODUCTION

Mean dimension was introduced by Gromov [11] about a decade ago, as an analogue of dimension for dynamical systems, and was studied systematically by Lindenstrauss and Weiss [20] for continuous actions of countable amenable groups on compact metrizable spaces. Among many beautiful results they obtained, it is especially notable that they used mean dimension to show that there exists a minimal action of  $\mathbb{Z}$  on some compact metrizable space which can not be embedded into  $[0, 1]^{\mathbb{Z}}$  equipped with the shift  $\mathbb{Z}$ -action in any way commuting with the  $\mathbb{Z}$ -actions. Mean dimension is further explored in [5, 6, 12, 16, 17, 19].

The notion of sofic groups was also introduced by Gromov [10] around the same time. The class of sofic groups include all discrete amenable groups and residually finite groups, and it is still an open question whether every group is sofic. For some nice exposition on sofic groups, see [4, 7, 8, 23, 26]. Using the idea of counting sofic approximations, in [2] Bowen defined entropy for measure-preserving actions of countable sofic groups on probability measure spaces, when there exists a countable generating partition with finite Shannon entropy. Together with David Kerr, in [14, 15] we extended Bowen's measure entropy to all measure-preserving actions of countable sofic groups on standard probability measure spaces, and defined topological entropy for continuous actions of countable sofic groups on compact metrizable spaces. The sofic measure entropy and sofic topological entropy are related by the variational principle [14]. Furthermore, the sofic entropies coincide with the classical entropies when the group is amenable [3, 15].

The goal of this article is to extend the mean dimension to continuous actions of countable sofic groups  $G$  on compact metrizable spaces  $X$ . In order to define sofic mean dimension, we use some approximate actions of  $G$  on finite sets as models,

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and replace  $X$  by certain spaces of approximately  $G$ -equivariant maps from the finite sets to  $X$ , which appeared first in the definition of sofic topological entropy [15]. A novelty here is that we replace open covers of  $X$  by certain open covers on these map spaces.

Lindenstrauss and Weiss studied two kinds of mean dimensions for actions of countable amenable groups in [20], one is topological, as the analogue of the covering dimension, and the other is metric, as the analogue of the lower box dimension.

We define the sofic mean topological dimension and establish some basic properties in Section 2, and show that it coincides with the Gromov-Lindenstrauss-Weiss mean topological dimension when the group is amenable in Section 3. Similarly, we discuss the sofic mean metric dimension in Sections 4 and 5. It is shown in Section 6 that the sofic mean topological dimension is always bounded above by the sofic mean metric dimension. We calculate the sofic mean dimensions for some Bernoulli shifts and show that every non-trivial factor of the shift action of  $G$  on  $[0, 1]^G$  has positive sofic mean dimensions in Section 7. In the last section, we show that actions with small-boundary property have zero sofic mean topological dimension.

To round up this section, we fix some notation. Throughout this paper,  $G$  will be a countable sofic group with identity element  $e_G$ . For  $d \in \mathbb{N}$ , we write  $[d]$  for the set  $\{1, \dots, d\}$  and  $\text{Sym}(d)$  for the permutation group of  $[d]$ . We fix a *sofic approximation sequence*  $\Sigma = \{\sigma_i : G \rightarrow \text{Sym}(d_i)\}_{i=1}^\infty$  for  $G$ , namely the following three conditions are satisfied:

- (1) for any  $s, t \in G$ , one has  $\lim_{i \rightarrow \infty} \frac{|\{a \in [d_i] : \sigma_i(s)\sigma_i(t)(a) = \sigma_i(st)(a)\}|}{d_i} = 1$ ;
- (2) for any distinct  $s, t \in G$ , one has  $\lim_{i \rightarrow \infty} \frac{|\{a \in [d_i] : \sigma_i(s)(a) = \sigma_i(t)(a)\}|}{d_i} = 0$ ;
- (3)  $\lim_{i \rightarrow \infty} d_i = +\infty$ .

The existence of such a sequence is equivalent to the condition that the countable group  $G$  is sofic. Note that the condition (1) and (2) imply the condition (3) when  $G$  is infinite.

For a map  $\sigma$  from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ , we write  $\sigma(s)(a)$  as  $\sigma_s(a)$  or  $sa$ , when there is no confusion. We say that  $\sigma$  is a good enough sofic approximation for  $G$  if for some large finite subset  $F$  of  $G$  which will be clear from the context, one has  $\frac{|\{a \in [d] : \sigma(s)\sigma(t)(a) = \sigma(st)(a)\}|}{d}$  very close to 1 for all  $s, t \in F$  and  $\frac{|\{a \in [d] : \sigma(s)(a) = \sigma(t)(a)\}|}{d}$  very close to 0 for all distinct  $s, t \in F$ .

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## 2. SOFIC MEAN TOPOLOGICAL DIMENSION

In this section we define the sofic mean topological dimension and establish some basic properties.

We start with recalling the definitions of covering dimension of compact metrizable spaces and mean topological dimension for actions of countable amenable groups.

For a compact space  $Y$  and two finite open covers  $\mathcal{U}$  and  $\mathcal{V}$  of  $Y$ , we say that  $\mathcal{V}$  *refines*  $\mathcal{U}$ , and write  $\mathcal{V} \succ \mathcal{U}$ , if every item of  $\mathcal{V}$  is contained in some item of  $\mathcal{U}$ .

**Definition 2.1.** Let  $Y$  be a compact space and  $\mathcal{U}$  a finite open cover of  $Y$ . We denote

$$\text{ord}(\mathcal{U}) = \max_{y \in Y} \sum_{U \in \mathcal{U}} 1_U(y) - 1, \text{ and } \mathcal{D}(\mathcal{U}) = \inf_{\mathcal{V} \succ \mathcal{U}} \text{ord}(\mathcal{V}),$$

where  $\mathcal{V}$  ranges over finite open covers of  $Y$  refining  $\mathcal{U}$ .

For a compact metrizable space  $X$ , its (*covering*) *dimension*  $\dim X$  is defined as  $\sup_{\mathcal{U}} \mathcal{D}(\mathcal{U})$  for  $\mathcal{U}$  ranging over finite open covers of  $X$ .

Let a countable amenable (discrete) group  $G$  act continuously on a compact metrizable space  $X$ . Let  $\mathcal{U}$  be a finite open cover of  $X$ . For a nonempty finite subset  $F$  of  $G$ , we set  $\mathcal{U}^F = \bigvee_{s \in F} s^{-1}\mathcal{U}$ . The function  $F \mapsto \mathcal{D}(\mathcal{U}^F)$  defined on the set of nonempty finite subsets of  $G$  satisfies the conditions of the Ornstein-Weiss lemma [22] [20, Theorem 6.1], thus  $\frac{\mathcal{D}(\mathcal{U}^F)}{|F|}$  converges to some real number, denoted by  $\text{mdim}(\mathcal{U})$ , when  $F$  becomes more and more left invariant. That is, for any  $\varepsilon > 0$ , there exist a nonempty finite subset  $K$  of  $G$  and  $\delta > 0$  such that  $|\frac{\mathcal{D}(\mathcal{U}^F)}{|F|} - \text{mdim}(\mathcal{U})| < \varepsilon$  for every nonempty finite subset  $F$  of  $G$  satisfying  $|KF \setminus F| < \delta|F|$ . The *mean topological dimension* of  $X$  [20, page 13] is defined as

$$\text{mdim}(X) = \sup_{\mathcal{U}} \text{mdim}(\mathcal{U}),$$

where  $\mathcal{U}$  ranges over finite open covers of  $X$ .

Throughout the rest of this section, we fix a countable sofic group  $G$  and a sofic approximation sequence  $\Sigma = \{\sigma_i : G \rightarrow \text{Sym}(d_i)\}_{i=1}^\infty$  for  $G$ , as defined in Section 1. Let  $\alpha$  be a continuous action of  $G$  on a compact metrizable space  $X$ .

Let  $\rho$  be a continuous pseudometric on  $X$ . For a given  $d \in \mathbb{N}$ , we define on the set of all maps from  $[d]$  to  $X$  the pseudometrics

$$\begin{aligned} \rho_2(\varphi, \psi) &= \left( \frac{1}{d} \sum_{a \in [d]} (\rho(\varphi(a), \psi(a)))^2 \right)^{1/2}, \\ \rho_\infty(\varphi, \psi) &= \max_{a \in [d]} \rho(\varphi(a), \psi(a)). \end{aligned}$$

**Definition 2.2.** Let  $F$  be a nonempty finite subset of  $G$  and  $\delta > 0$ . Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . We define  $\text{Map}(\rho, F, \delta, \sigma)$  to be the set of all maps  $\varphi : [d] \rightarrow X$  such that  $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) \leq \delta$  for all  $s \in F$ .

The space  $\text{Map}(\rho, F, \delta, \sigma)$  appeared first in [15, Section 2], and was used to define the topological entropy of the action  $\alpha$ . Eventually we shall take  $\sigma$  to be  $\sigma_i$  for large  $i$ . Then the condition (1) in the requirements of  $\Sigma$  says that  $\sigma$  is approximately a group homomorphism of  $G$  into  $\text{Sym}(d)$ , and then we can think of  $\sigma$  as

an approximate action of  $G$  on  $[d]$ . The space  $\text{Map}(\rho, F, \delta, \sigma)$  is the set of approximately  $G$ -equivariant maps from  $[d]$  into  $X$ . When  $G$  is amenable and  $\sigma$  comes from some Følner set of  $G$ , there is a natural map from  $X$  to  $\text{Map}(\rho, F, \delta, \sigma)$ , as clear in the proof of Theorem 3.1 below. For general sofic group  $G$ , we shall replace  $X$  by  $\text{Map}(\rho, F, \delta, \sigma)$  when defining invariants of  $\alpha$ .

For a finite open cover  $\mathcal{U}$  of  $X$ , we denote by  $\mathcal{U}^d$  the finite open cover of  $X^{[d]}$  consisting of  $U_1 \times U_2 \times \cdots \times U_d$  for  $U_1, \dots, U_d \in \mathcal{U}$ . Note that  $\text{Map}(\rho, F, \delta, \sigma)$  is a closed subset of  $X^{[d]}$ . We shall replace  $\mathcal{U}^F$  in the amenable group case by the restriction  $\mathcal{U}^d|_{\text{Map}(\rho, F, \delta, \sigma)}$  of  $\mathcal{U}^d$  to  $\text{Map}(\rho, F, \delta, \sigma)$ . Denote  $\mathcal{D}(\mathcal{U}^d|_{\text{Map}(\rho, F, \delta, \sigma)})$  by  $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)$ .

**Definition 2.3.** Let  $\rho$  be a compatible metric on  $X$ . Let  $F$  be a nonempty finite subset of  $G$  and  $\delta > 0$ . For a finite open cover  $\mathcal{U}$  of  $X$  we define

$$\begin{aligned}\mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta) &= \overline{\lim}_{i \rightarrow \infty} \frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma_i)}{d_i}, \\ \mathcal{D}_\Sigma(\mathcal{U}, \rho, F) &= \inf_{\delta > 0} \mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta), \\ \mathcal{D}_\Sigma(\mathcal{U}, \rho) &= \inf_F \mathcal{D}_\Sigma(\mathcal{U}, \rho, F),\end{aligned}$$

where  $F$  in the third line ranges over the nonempty finite subsets of  $G$ . If  $\text{Map}(\rho, F, \delta, \sigma_i)$  is empty for all sufficiently large  $i$ , we set  $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta) = -\infty$ . We define the *sofic mean topological dimension* of  $\alpha$  as

$$\text{mdim}_\Sigma(X, \rho) = \sup_{\mathcal{U}} \mathcal{D}_\Sigma(\mathcal{U}, \rho)$$

for  $\mathcal{U}$  ranging over finite open covers of  $X$ . As shown by Lemma 2.4 below, the quantities  $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F)$ ,  $\mathcal{D}_\Sigma(\mathcal{U}, \rho)$  and  $\text{mdim}_\Sigma(X, \rho)$  do not depend on the choice of  $\rho$ , and we shall write them as  $\mathcal{D}_\Sigma(\mathcal{U}, F)$ ,  $\mathcal{D}_\Sigma(\mathcal{U})$  and  $\text{mdim}_\Sigma(X)$  respectively.

**Lemma 2.4.** *Let  $\rho$  and  $\rho'$  be compatible metrics on  $X$ . For any nonempty finite subset  $F$  of  $G$  and any finite open cover  $\mathcal{U}$  of  $X$ , one has  $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F) = \mathcal{D}_\Sigma(\mathcal{U}, \rho', F)$ .*

*Proof.* By symmetry it suffices to show  $\mathcal{D}_\Sigma(\mathcal{U}, \rho, F) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho', F)$ . Let  $\delta > 0$ .

Take  $\delta' > 0$  be a small positive number which we shall determine in a moment. We claim that for any map  $\sigma$  from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$  one has  $\text{Map}(\rho, F, \delta', \sigma) \subseteq \text{Map}(\rho', F, \delta, \sigma)$ . Let  $\varphi \in \text{Map}(\rho, F, \delta', \sigma)$ . For each  $s \in F$ , set

$$\mathcal{W}_s = \{a \in [d] : \rho(s\varphi(a), \varphi(sa)) \leq \sqrt{\delta'}\}.$$

Since  $\rho_2(\alpha_s \circ \varphi, \varphi \circ \sigma_s) \leq \delta'$ , one has  $|\mathcal{W}_s| \geq (1 - \delta')d$ . Taking  $\delta'$  small enough, we may assume that for any  $x, y \in X$  with  $\rho(x, y) \leq \sqrt{\delta'}$ , one has  $\rho'(x, y) \leq \delta/2$ . Then one has

$$(\rho'_2(\alpha_s \circ \varphi, \varphi \circ \sigma_s))^2 \leq \frac{|\mathcal{W}_s|}{d} \cdot \frac{\delta^2}{4} + (1 - \frac{|\mathcal{W}_s|}{d})(\text{diam}(X, \rho'))^2$$

$$\leq \frac{\delta^2}{4} + \delta'(\text{diam}(X, \rho'))^2 \leq \delta^2,$$

granted that  $\delta'$  is small enough. Therefore  $\varphi \in \text{Map}(\rho', F, \delta, \sigma)$ . This proves the claim.

Since  $\text{Map}(\rho, F, \delta', \sigma) \subseteq \text{Map}(\rho', F, \delta, \sigma)$ , clearly  $\mathcal{D}(\mathcal{U}, \rho, F, \delta', \sigma) \leq \mathcal{D}(\mathcal{U}, \rho', F, \delta, \sigma)$ . Thus  $\mathcal{D}(\mathcal{U}, \rho, F) \leq \mathcal{D}(\mathcal{U}, \rho, F, \delta') \leq \mathcal{D}(\mathcal{U}, \rho', F, \delta)$ . Letting  $\delta \rightarrow 0$ , we get  $\mathcal{D}(\mathcal{U}, \rho, F) \leq \mathcal{D}(\mathcal{U}, \rho', F)$  as desired.  $\square$

Lindenstrauss and Weiss established the next two propositions in the case  $G$  is amenable [20, page 5, Proposition 2.8].

**Proposition 2.5.** *Let  $G$  act continuously on a compact metrizable space  $X$ . Let  $Y$  be a closed  $G$ -invariant subset of  $X$ . Then  $\text{mdim}_\Sigma(Y) \leq \text{mdim}_\Sigma(X)$ .*

*Proof.* Let  $\rho$  be a compatible metric on  $X$ . Then  $\rho$  restricts to a compatible metric  $\rho'$  on  $Y$ .

Let  $\mathcal{U}$  be a finite open cover of  $Y$ . Then we can find a finite open cover  $\mathcal{V}$  of  $X$  such that  $\mathcal{U}$  is the restriction of  $\mathcal{V}$  to  $Y$ . Note that  $\text{Map}(\rho', F, \delta, \sigma) \subseteq \text{Map}(\rho, F, \delta, \sigma)$  for any nonempty finite subset  $F$  of  $G$ , any  $\delta > 0$ , and any map  $\sigma$  from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . Furthermore, the restriction of  $\mathcal{V}^d|_{\text{Map}(\rho, F, \delta, \sigma)}$  on  $\text{Map}(\rho', F, \delta, \sigma)$  is exactly  $\mathcal{U}^d|_{\text{Map}(\rho', F, \delta, \sigma)}$ . Thus  $\mathcal{D}(\mathcal{U}, \rho', F, \delta, \sigma) \leq \mathcal{D}(\mathcal{V}, \rho, F, \delta, \sigma)$ . It follows that  $\text{mdim}_\Sigma(\mathcal{U}) \leq \text{mdim}_\Sigma(\mathcal{V}) \leq \text{mdim}_\Sigma(X)$ . Since  $\mathcal{U}$  is an arbitrary finite open cover of  $Y$ , we get  $\text{mdim}_\Sigma(Y) \leq \text{mdim}_\Sigma(X)$ .  $\square$

**Proposition 2.6.** *Let  $G$  act continuously on a compact metrizable space  $X_n$  for each  $1 \leq n < R$ , where  $R \in \mathbb{N} \cup \{\infty\}$ . Consider the product action of  $G$  on  $X := \prod_{1 \leq n < R} X_n$ . Then  $\text{mdim}_\Sigma(X) \leq \sum_{1 \leq n < R} \text{mdim}_\Sigma(X_n)$ .*

*Proof.* Let  $\rho$  and  $\rho^{(n)}$  be compatible metrics on  $X$  and  $X_n$  respectively. Denote by  $\pi_n$  the projection of  $X$  onto  $X_n$ . Let  $\mathcal{U}$  be a finite open cover of  $X$ . Then there are an  $N \in \mathbb{N}$  with  $N < R$  and a finite open cover  $\mathcal{V}_n$  of  $X_n$  for all  $1 \leq n \leq N$  such that

$$\mathcal{V} := \bigvee_{n=1}^N \pi_n^{-1}(\mathcal{V}_n) \succ \mathcal{U}.$$

Let  $F$  be a nonempty finite subset of  $G$  and  $\delta > 0$ . Then we can find  $\delta' > 0$  such that for any map  $\sigma$  from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$  and any  $\varphi \in \text{Map}(\rho, F, \delta', \sigma)$  one has  $\pi_n \circ \varphi \in \text{Map}(\rho^{(n)}, F, \delta, \sigma)$  for all  $1 \leq n \leq N$ . It follows that we have a continuous map  $\Phi_n : \text{Map}(\rho, F, \delta', \sigma) \rightarrow \text{Map}(\rho^{(n)}, F, \delta, \sigma)$  sending  $\varphi$  to  $\pi_n \circ \varphi$  for each  $1 \leq n \leq N$ . Note that

$$\mathcal{V}^d|_{\text{Map}(\rho, F, \delta', \sigma)} = \bigvee_{n=1}^N \Phi_n^{-1}(\mathcal{V}_n^d|_{\text{Map}(\rho^{(n)}, F, \delta, \sigma)}).$$

For any finite open covers  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of a compact metrizable space  $Y$  one has  $\mathcal{D}(\mathcal{U}_1 \vee \mathcal{U}_2) \leq \mathcal{D}(\mathcal{U}_1) + \mathcal{D}(\mathcal{U}_2)$  [20, Corollary 2.5]. Thus

$$\mathcal{D}(\mathcal{U}, \rho, F, \delta', \sigma) \leq \mathcal{D}(\mathcal{V}, \rho, F, \delta', \sigma) \leq \sum_{n=1}^N \mathcal{D}(\mathcal{V}_n, \rho^{(n)}, F, \delta, \sigma),$$

and hence  $\mathcal{D}_\Sigma(\mathcal{U}, \rho) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho, F, \delta') \leq \sum_{n=1}^N \mathcal{D}_\Sigma(\mathcal{V}_n, \rho^{(n)}, F, \delta)$ . Since  $F$  and  $\delta$  are arbitrary, we get

$$\mathcal{D}_\Sigma(\mathcal{U}, \rho) \leq \sum_{n=1}^N \mathcal{D}_\Sigma(\mathcal{V}_n, \rho^{(n)}) \leq \sum_{n=1}^N \text{mdim}_\Sigma(X_n) \leq \sum_{1 \leq n < R} \text{mdim}_\Sigma(X_n).$$

Therefore  $\text{mdim}_\Sigma(X) \leq \sum_{1 \leq n < R} \text{mdim}_\Sigma(X_n)$  as desired.  $\square$

If a property  $P$  for continuous  $G$ -actions on compact Hausdorff spaces is preserved by products, subsystems, and isomorphisms, then for any continuous  $G$ -action on a compact Hausdorff space  $X$ , there is a largest factor  $Y$  of  $X$  with property  $P$  [9, Proposition 2.9.1]. We extend this fact to actions on compact metrizable spaces, the proof of which is implicit in the proof of [19, Proposition 6.12].

**Lemma 2.7.** *Let  $\Gamma$  be a topological group. Let  $P$  be a property for continuous  $\Gamma$ -actions on compact metrizable spaces. Suppose that  $P$  is preserved by countable products, subsystems, and isomorphisms. Then any continuous  $\Gamma$ -action on a compact metrizable space  $X$  has a largest factor  $Y$  with property  $P$ , i.e. for any factor  $Z$  of  $X$  with property  $P$  there is a unique ( $\Gamma$ -equivariant continuous surjective) map  $Y \rightarrow Z$  making the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \downarrow \\ & & Z \end{array}$$

*commute.*

*Proof.* For each factor  $Z$  of  $X$  with factor map  $\pi_Z : X \rightarrow Z$ , denote by  $R_Z$  the closed subset  $\{(x, y) \in X^2 : \pi_Z(x) = \pi_Z(y)\}$  of  $X^2$ . Denote by  $R$  the set  $\bigcap_Z R_Z$  for  $Z$  ranging over factors of  $X$  with property  $P$ . Since  $X$  is compact metrizable, we can find factors  $Z_1, Z_2, \dots$  of  $X$  with property  $P$  such that  $\bigcap_{n=1}^\infty R_{Z_n} = R$ . Consider the map  $\pi : X \rightarrow \prod_{n=1}^\infty Z_n$  sending  $x$  to  $(\pi_{Z_n}(x))_{n=1}^\infty$ . Then  $Y := \pi(X)$  is a closed  $\Gamma$ -invariant subset of  $\prod_{n=1}^\infty Z_n$  and is a factor of  $X$ . Furthermore,  $R_Y = R$ . By the assumption on  $P$  and  $Z_n$ , we see that the  $\Gamma$ -action on  $Y$  has property  $P$ . Since  $R_Z \supseteq R = R_Y$  for every factor  $Z$  of  $X$  with property  $P$ , clearly  $Y$  is the largest factor of  $X$  with property  $P$ .  $\square$

Note that if  $G$  acts continuously on a compact metrizable space  $X$  with  $\text{mdim}_\Sigma(X) \geq 0$ , then  $\text{mdim}_\Sigma(Y) \geq 0$  for every factor  $Y$  of  $X$ . From Lemma 2.7 and Propositions 2.5 and 2.6, taking property  $P$  to be having sofic mean topological dimension

at most 0, we obtain the following result, which was established by Lindenstrauss for countable amenable groups [19, Proposition 6.12].

**Proposition 2.8.** *Let  $G$  act continuously on a compact metrizable space  $X$  with  $\text{mdim}_\Sigma(X) \geq 0$ . Then  $X$  has a largest factor  $Y$  satisfying  $\text{mdim}_\Sigma(Y) = 0$ .*

### 3. SOFIC MEAN TOPOLOGICAL DIMENSION FOR AMENABLE GROUPS

In this section we show that the sofic mean topological dimension extends the mean topological dimension for actions of countably infinite amenable groups:

**Theorem 3.1.** *Let a countably infinite (discrete) amenable group  $G$  act continuously on a compact metrizable space  $X$ . Let  $\Sigma$  to a sofic approximation sequence of  $G$ . Then*

$$\text{mdim}_\Sigma(X) = \text{mdim}(X).$$

Theorem 3.1 follows directly from Lemmas 3.3 and 3.5 below.

We need the following Rokhlin lemma several times.

**Lemma 3.2.** [15, Lemma 4.6] *Let  $G$  be a countable amenable group. Let  $0 \leq \tau < 1$ ,  $0 < \eta < 1$ ,  $K$  be a nonempty finite subset of  $G$ , and  $\delta > 0$ . Then there are an  $\ell \in \mathbb{N}$ , nonempty finite subsets  $F_1, \dots, F_\ell$  of  $G$  with  $|KF_k \setminus F_k| < \delta|F_k|$  and  $|F_k K \setminus F_k| < \delta|F_k|$  for all  $k = 1, \dots, \ell$ , a finite set  $F \subseteq G$  containing  $e_G$ , and an  $\eta' > 0$  such that, for every  $d \in \mathbb{N}$ , every map  $\sigma : G \rightarrow \text{Sym}(d)$  for which there is a set  $\mathcal{B} \subseteq [d]$  satisfying  $|\mathcal{B}| \geq (1 - \eta')d$  and*

$$\sigma_{st}(a) = \sigma_s \sigma_t(a), \sigma_s(a) \neq \sigma_{s'}(a), \sigma_{e_G}(a) = a$$

*for all  $a \in \mathcal{B}$  and  $s, t, s' \in F$  with  $s \neq s'$ , and every set  $\mathcal{W} \subseteq [d]$  with  $|\mathcal{W}| \geq (1 - \tau)d$ , there exist  $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$  such that*

- (1) *for every  $k = 1, \dots, \ell$ , the map  $(s, c) \mapsto \sigma_s(c)$  from  $F_k \times \mathcal{C}_k$  to  $\sigma(F_k)\mathcal{C}_k$  is bijective,*
- (2) *the sets  $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$  are pairwise disjoint and  $|\bigcup_{k=1}^\ell \sigma(F_k)\mathcal{C}_k| \geq (1 - \tau - \eta)d$ .*

**Lemma 3.3.** *Let a countably infinite amenable group  $G$  act continuously on a compact metrizable space  $X$ . Then for any finite open cover  $\mathcal{U}$  of  $X$  we have  $\mathcal{D}_\Sigma(\mathcal{U}) \geq \text{mdim}(\mathcal{U})$ . In particular,  $\text{mdim}_\Sigma(X) \geq \text{mdim}(X)$ .*

*Proof.* It suffices to show that  $\mathcal{D}_\Sigma(\mathcal{U}) \geq \text{mdim}(\mathcal{U}) - 2\theta$  for every  $\theta > 0$ .

Fix a compatible metric  $\rho$  on  $X$ . Let  $F$  be a nonempty finite subset of  $G$  and  $\delta > 0$ . Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . Now it suffices to show that if  $\sigma$  is a good enough sofic approximation then

$$\frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)}{d} \geq \text{mdim}(\mathcal{U}) - 2\theta.$$

Take a finite subset  $K$  of  $G$  containing  $F$  and  $\varepsilon > 0$  such that for any nonempty finite subset  $F'$  of  $G$  with  $|KF' \setminus F'| < \varepsilon|F'|$  one has

$$\frac{\mathcal{D}(\mathcal{U}^{F'})}{|F'|} \geq \text{mdim}(\mathcal{U}) - \theta.$$

Take  $0 < \delta' < 1$  such that  $\sqrt{\delta'} \text{diam}(X, \rho) < \delta/2$  and  $(\text{mdim}(\mathcal{U}) - \theta)(1 - \delta') \geq \text{mdim}(\mathcal{U}) - 2\theta$ . By Lemma 3.2 there are an  $\ell \in \mathbb{N}$  and nonempty finite subsets  $F_1, \dots, F_\ell$  of  $G$  satisfying  $|KF_k \setminus F_k| < \min(\varepsilon, \delta')|F_k|$  for all  $k = 1, \dots, \ell$  such that for every map  $\sigma : G \rightarrow \text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  and every  $\mathcal{W} \subseteq [d]$  with  $|\mathcal{W}| \geq (1 - \delta'/2)d$  there exist  $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$  satisfying the following:

- (1) for every  $k = 1, \dots, \ell$ , the map  $(s, c) \mapsto \sigma_s(c)$  from  $F_k \times \mathcal{C}_k$  to  $\sigma(F_k)\mathcal{C}_k$  is bijective,
- (2) the sets  $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$  are pairwise disjoint and  $|\bigcup_{k=1}^\ell \sigma(F_k)\mathcal{C}_k| \geq (1 - \delta')d$ .

Let  $\sigma : G \rightarrow \text{Sym}(d)$  for some  $d \in \mathbb{N}$  be a good enough sofic approximation for  $G$  such that  $|\mathcal{W}| \geq (1 - \delta'/2)d$  for

$$\mathcal{W} := \{a \in [d] : \sigma_t \sigma_s(a) = \sigma_{ts}(a) \text{ for all } t \in F, s \in \bigcup_{k=1}^\ell F_k\}.$$

Then we have  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  as above.

Since  $G$  is infinite, we can take a map  $\psi_k : \mathcal{C}_k \rightarrow G$  for all  $k = 1, \dots, \ell$  such that the map  $\Psi$  from  $\bigsqcup_{k=1}^\ell F_k \times \mathcal{C}_k$  to  $G$  sending  $(s, c) \in F_k \times \mathcal{C}_k$  to  $s\psi_k(c)$  is injective. Denote by  $\tilde{F}$  the range of  $\Psi$ . Note that  $|K\tilde{F} \setminus \tilde{F}| < \varepsilon|\tilde{F}|$ . Thus

$$\frac{\mathcal{D}(\mathcal{U}^{\tilde{F}})}{|\tilde{F}|} \geq \text{mdim}(\mathcal{U}) - \theta.$$

Pick  $x_0 \in X$ . For each  $x \in X$  define a map  $\varphi_x : [d] \rightarrow X$  by  $\varphi_x(a) = x_0$  for all  $a \in [d] \setminus \bigcup_{k=1}^\ell \sigma(F_k)\mathcal{C}_k$ , and

$$\varphi_x(sc) = s\psi_k(c)x$$

for all  $k \in \{1, \dots, \ell\}$ ,  $c \in \mathcal{C}_k$ , and  $s \in F_k$ . Note that if  $t \in F$ ,  $k \in \{1, \dots, \ell\}$ ,  $s \in F_k$ ,  $c \in \mathcal{C}_k$ , and  $ts \in F_k$ , then  $\sigma_t \sigma_s(c) = \sigma_{ts}(c)$ , and hence  $\alpha_t \circ \varphi_x(sc) = \varphi_x \circ \sigma_t(sc)$ . It follows that  $\rho_2(\alpha_t \circ \varphi_x, \varphi_x \circ \sigma_t) < \delta$  for all  $t \in F$ , and thus  $\varphi_x \in \text{Map}(\rho, F, \delta, \sigma)$ .

Note that the map  $\Phi$  from  $X$  to  $\text{Map}(\rho, F, \delta, \sigma)$  sending  $x$  to  $\varphi_x$  is continuous, and  $\Phi^{-1}(\mathcal{U}^d|_{\text{Map}(\rho, F, \delta, \sigma)}) = \mathcal{U}^{\tilde{F}}$ . Thus  $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) \geq \mathcal{D}(\mathcal{U}^{\tilde{F}})$ . Therefore

$$\frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)}{d} \geq \frac{\mathcal{D}(\mathcal{U}^{\tilde{F}})}{|\tilde{F}|} \cdot \frac{|\tilde{F}|}{d} \geq (\text{mdim}(\mathcal{U}) - \theta)(1 - \delta') \geq \text{mdim}(\mathcal{U}) - 2\theta,$$

as desired.  $\square$



Let  $\mathcal{U}$  be a finite open cover of a compact metrizable space  $X$ . A continuous map  $f$  from  $X$  into another compact metrizable space  $Y$  is said to be  $\mathcal{U}$ -compatible if for each  $y \in Y$ , the set  $f^{-1}(y)$  is contained in some  $U \in \mathcal{U}$  [20, Definition 2.2 and Proposition 2.3]. We need the following fact:

**Lemma 3.4.** [20, Proposition 2.4] *Let  $\mathcal{U}$  be a finite open cover of a compact metrizable space  $X$ , and  $k \geq 0$ . Then  $\mathcal{D}(\mathcal{U}) \leq k$  if and only if there is a  $\mathcal{U}$ -compatible continuous map  $f : X \rightarrow Y$  for some compact metrizable space  $Y$  with dimension  $k$ .*

**Lemma 3.5.** *Let a countable amenable group  $G$  act continuously on a compact metrizable space  $X$ . Then  $\text{mdim}_\Sigma(X) \leq \text{mdim}(X)$ .*

*Proof.* Fix a compatible metric  $\rho$  on  $X$ . Let  $\mathcal{U}$  be a finite open cover of  $X$ . It suffices to show that  $\mathcal{D}_\Sigma(\mathcal{U}) \leq \text{mdim}(X)$ .

Take a finite open cover  $\mathcal{V}$  of  $X$  such that for every  $V \in \mathcal{V}$ , one has  $\overline{V} \subseteq U$  for some  $U \in \mathcal{U}$ . Then it suffices to show  $\mathcal{D}_\Sigma(\mathcal{U}) \leq \text{mdim}(\mathcal{V}) + 3\theta$  for every  $\theta > 0$ . We can find  $\eta > 0$  such that for every  $V \in \mathcal{V}$ , one has  $B(V, \eta) = \{x \in X : \rho(x, V) < \eta\} \subseteq U$  for some  $U \in \mathcal{U}$ .

Take a nonempty finite subset  $K$  of  $G$  and  $\varepsilon > 0$  such that for any nonempty finite subset  $F'$  of  $G$  with  $|KF' \setminus F'| < \varepsilon|F'|$  one has

$$\frac{\mathcal{D}(\mathcal{V}^{F'})}{|F'|} \leq \text{mdim}(\mathcal{V}) + \theta.$$

Take  $\tau > 0$  with  $\tau\mathcal{D}(\mathcal{U}) \leq \theta$ . By Lemma 3.2 there are an  $\ell \in \mathbb{N}$  and nonempty finite subsets  $F_1, \dots, F_\ell$  of  $G$  satisfying  $|KF_k \setminus F_k| < \varepsilon|F_k|$  for all  $k = 1, \dots, \ell$  such that for every map  $\sigma : G \rightarrow \text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  and every  $\mathcal{W} \subseteq [d]$  with  $|\mathcal{W}| \geq (1 - \tau/2)d$  there exist  $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$  satisfying the following:

- (1) for every  $k = 1, \dots, \ell$ , the map  $(s, c) \mapsto \sigma_s(c)$  from  $F_k \times \mathcal{C}_k$  to  $\sigma(F_k)\mathcal{C}_k$  is bijective,
- (2) the sets  $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$  are pairwise disjoint and  $|\bigcup_{k=1}^\ell \sigma(F_k)\mathcal{C}_k| \geq (1 - \tau)d$ .

Set  $F = \bigcup_{k=1}^\ell F_k^{-1}$ . Take  $\kappa > 0$  such that for any  $x, y \in X$  with  $\rho(x, y) \leq \kappa$  one has  $\rho(s^{-1}x, s^{-1}y) < \eta$  for all  $s \in F$ . Take  $\delta > 0$  with  $\delta^{1/2} < \kappa$  and  $\delta|\mathcal{U}||F| \leq \theta$ .

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . Now it suffices to show that if  $\sigma$  is a good enough sofic approximation then

$$\frac{\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma)}{d} \leq \text{mdim}(\mathcal{V}) + 3\theta.$$

Denote by  $\mathcal{W}$  the subset of  $[d]$  consisting of  $a$  satisfying  $\sigma_s \sigma_{s^{-1}}(a) = \sigma_{e_G}(a) = a$  for all  $s \in F$ . Assuming that  $\sigma$  is a good enough sofic approximation, we have  $|\mathcal{W}| \geq (1 - \tau/2)d$  and can find  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  as above. Set  $\mathcal{Z} = [d] \setminus \bigcup_{k=1}^\ell \sigma(F_k)\mathcal{C}_k$ . Then  $|\mathcal{Z}| \leq \tau d$ .

For every  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$ , we have  $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) \leq \delta$  for all  $s \in F$ . Thus the set  $\Lambda_\varphi$  of all  $a \in [d]$  satisfying

$$\rho(\varphi(sa), s\varphi(a)) \leq \delta^{1/2}$$

for all  $s \in F$  has cardinality at least  $(1 - |F|\delta)d$ .

Take a partition of unity  $\{\zeta_U\}_{U \in \mathcal{U}}$  for  $X$  subordinate to  $\mathcal{U}$ . That is, each  $\zeta_U$  is a continuous function  $X \rightarrow [0, 1]$  with support contained in  $U$ , and

$$\sum_{U \in \mathcal{U}} \zeta_U = 1.$$

Define a continuous map  $\xi : X \rightarrow [0, 1]^{\mathcal{U}}$  by  $\xi(x)_U = \zeta_U(x)$  for  $x \in X$  and  $U \in \mathcal{U}$ . Consider the continuous map  $h : \text{Map}(\rho, F, \delta, \sigma) \rightarrow ([0, 1]^{\mathcal{U}})^{[d]}$  defined by

$$h(\varphi)_a = \xi(\varphi(a)) \max(\max_{s \in F} \rho(s\varphi(a), \varphi(sa)) - \kappa, 0)$$

for  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$  and  $a \in [d]$ . Denote by  $\nu$  the point of  $[0, 1]^{\mathcal{U}}$  having all coordinates 0. Set  $X_0$  to be the subset of  $([0, 1]^{\mathcal{U}})^{[d]}$  consisting of elements whose coordinates are equal to  $\nu$  at at least  $(1 - |F|\delta)d$  elements of  $[d]$ . For each  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$ , note that  $h(\varphi)_a = \nu$  for all  $a \in \Lambda_\varphi$  by our choice of  $\delta$  and hence  $h(\varphi) \in X_0$ . Thus we may think of  $h$  as a map from  $\text{Map}(\rho, F, \delta, \sigma)$  into  $X_0$ . Since the union of finitely many closed subsets of dimension at most  $m$  has dimension at most  $m$  [13, page 30 and Theorem V.8], we get  $\dim X_0 \leq |\mathcal{U}||F|\delta d \leq \theta d$ .

For each  $1 \leq k \leq \ell$ , by Lemma 3.4 we can find a compact metrizable space  $Y_k$  with  $\dim Y_k \leq \mathcal{D}(\mathcal{V}^{F_k})$  and a  $\mathcal{V}^{F_k}$ -compatible continuous map  $f_k : X \rightarrow Y_k$ .

By Lemma 3.4 we can find a compact metrizable space  $Z$  with  $\dim Z \leq \mathcal{D}(\mathcal{U})$  and a  $\mathcal{U}$ -compatible continuous map  $g : X \rightarrow Z$ .

Now define a continuous map  $\Psi : \text{Map}(\rho, F, \delta, \sigma) \rightarrow X_0 \times (\prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k) \times (\prod_{a \in \mathcal{Z}} Z)$  as follows. For  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$ , the coordinate of  $\Psi(\varphi)$  in  $X_0$  is  $h(\varphi)$ , in  $Y_k$  for  $1 \leq k \leq \ell$  and  $c \in \mathcal{C}_k$  is  $f_k(\varphi(c))$ , in  $Z$  for  $a \in \mathcal{Z}$  is  $g(\varphi(a))$ . We claim that  $\Psi$  is  $\mathcal{U}^d|_{\text{Map}(\rho, F, \delta, \sigma)}$ -compatible. Let  $w \in X_0 \times (\prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k) \times (\prod_{a \in \mathcal{Z}} Z)$ . We need to show that for each  $a \in [d]$  there is some  $U \in \mathcal{U}$  depending only on  $w$  and  $a$  such that  $\varphi(a) \in U$  for every  $\varphi \in \Psi^{-1}(w)$ . We write the coordinates of  $w$  in  $X_0$ ,  $\prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k$ , and  $\prod_{a \in \mathcal{Z}} Z$  as  $w^1$ ,  $w^2$ , and  $w^3$  respectively.

For each  $a \in \mathcal{Z}$ , since  $g$  is  $\mathcal{U}$ -compatible, one has  $g^{-1}(w_a^3) \subseteq U_{w_a^3}$  for some  $U_{w_a^3} \in \mathcal{U}$ . Then  $\varphi(a) \in U_{w_a^3}$  for every  $\varphi \in \Psi^{-1}(w)$  and  $a \in \mathcal{Z}$ .

For every  $1 \leq k \leq \ell$  and  $c \in \mathcal{C}_k$ , since  $f_k$  is  $\mathcal{V}^{F_k}$ -compatible, one has  $f_k^{-1}(w_{k,c}^2) \subseteq \bigcap_{s^{-1} \in F_k} sV_{k,c,s}$  for some  $V_{k,c,s} \in \mathcal{V}$  for every  $s^{-1} \in F_k$ . By the choice of  $\eta$ ,  $B(V_{k,c,s}, \eta)$  is contained in some  $U_{k,c,s} \in \mathcal{U}$ . For every  $a \in [d] \setminus \mathcal{Z}$ , we distinguish the two cases  $w_a^1 \neq \nu$  and  $w_a^1 = \nu$ . If  $w_a^1 \neq \nu$ , then  $(w_a^1)_U \neq 0$  for some  $U \in \mathcal{U}$ , and then for  $\varphi \in \Psi^{-1}(w)$  one has  $\zeta_U(\varphi(a)) > 0$  and hence  $\varphi(a) \in U$ . Suppose that  $w_a^1 = \nu$ . Say,  $a = \sigma(s^{-1})c$  for some  $1 \leq k \leq \ell$ ,  $s^{-1} \in F_k$  and  $c \in \mathcal{C}_k$ . Let  $\varphi \in \Psi^{-1}(w)$ . Since  $c \in \mathcal{C}_k \subseteq \mathcal{W}$  and  $s \in F$ , one has  $sa = \sigma_s \sigma_{s^{-1}}(c) = c$ . As  $\{\zeta_U\}_{U \in \mathcal{U}}$  is a partition of

unity of  $X$ ,  $\xi(\varphi(a)) \neq \nu$ . But  $h(\varphi)_a = w_a^1 = \nu$ . Thus  $\max_{s' \in F} \rho(s'\varphi(a), \varphi(s'a)) \leq \kappa$ . In particular, one has  $\rho(s\varphi(a), \varphi(c)) = \rho(s\varphi(a), \varphi(sa)) \leq \kappa$ . From our choice of  $\kappa$ , one gets  $\rho(\varphi(a), s^{-1}\varphi(c)) < \eta$ . Since  $f_k(\varphi(c)) = w_{k,c}^2$ , we have  $\varphi(c) \in sV_{k,c,s}$  and hence  $s^{-1}\varphi(c) \in V_{k,c,s}$ . Therefore  $\varphi(a) \in U_{k,c,s}$ . This proves the claim.

From Lemma 3.4 we get

$$\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) \leq \dim X_0 \times \left( \prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k \right) \times \left( \prod_{a \in \mathcal{Z}} Z \right).$$

Since the dimension of the product of two compact metrizable spaces is at most the sum of the dimensions of the factors [13, page 33 and Theorem V.8], we have

$$\begin{aligned} \dim X_0 \times \left( \prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} Y_k \right) \times \left( \prod_{a \in \mathcal{Z}} Z \right) &\leq \dim X_0 + \sum_{k=1}^{\ell} |\mathcal{C}_k| \dim Y_k + |\mathcal{Z}| \dim Z \\ &\leq \theta d + \sum_{k=1}^{\ell} |\mathcal{C}_k| \mathcal{D}(\mathcal{V}^{F_k}) + |\mathcal{Z}| \mathcal{D}(\mathcal{U}) \\ &\leq \theta d + \sum_{k=1}^{\ell} |\mathcal{C}_k| |F_k| (\text{mdim}(\mathcal{V}) + \theta) + \tau d \mathcal{D}(\mathcal{U}) \\ &\leq \theta d + d(\text{mdim}(\mathcal{V}) + \theta) + \theta d \\ &= d(\text{mdim}(\mathcal{V}) + 3\theta). \end{aligned}$$

Therefore  $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) \leq d(\text{mdim}(\mathcal{V}) + 3\theta)$  as desired.  $\square$

**Remark 3.6.** Theorem 3.1 fails when  $G$  is finite. Indeed, when a finite group  $G$  acts continuously on a compact metrizable space  $X$ , one has  $\text{mdim}(X) = \frac{1}{|G|} \dim X$ . There are compact metrizable finite-dimensional spaces  $X$  satisfying  $\dim X^2 < 2 \dim X$  (see [1]). For such  $X$ , Lemma 3.7 below implies that  $\text{mdim}_{\Sigma}(X) < \text{mdim}(X)$ .

If  $X$  is a compact metrizable space with finite dimension, then for any  $n, m \in \mathbb{N}$  one has  $\dim X^n \times X^m \leq \dim X^n + \dim X^m$  [13, page 33 and Theorem V.8] and hence  $\frac{\dim X^n}{n} \rightarrow \inf_{m \in \mathbb{N}} \frac{\dim X^m}{m}$  as  $n \rightarrow \infty$ .

**Lemma 3.7.** *Let a finite group  $G$  act continuously on a compact metrizable finite-dimensional space  $X$ . Then  $\text{mdim}_{\Sigma}(X) \leq \frac{1}{|G|} \inf_{m \in \mathbb{N}} \frac{\dim X^m}{m}$ .*

*Proof.* The proof is similar to that of Lemma 3.5. Fix a compatible metric  $\rho$  on  $X$ . Set  $\lambda = \frac{1}{|G|} \inf_{m \in \mathbb{N}} \frac{\dim X^m}{m}$ . Let  $\mathcal{U}$  be a finite open cover of  $X$  and  $\theta > 0$ . It suffices to show that  $\mathcal{D}_{\Sigma}(\mathcal{U}) \leq \lambda + 3\theta$ .

We can find  $\eta > 0$  such that for every  $y \in X$ , one has  $\{x \in X : \rho(x, y) < \eta\} \subseteq U$  for some  $U \in \mathcal{U}$ .

Take  $M > 0$  such that  $\frac{\dim X^m}{m} \leq (\lambda + \theta)|G|$  for all  $m \geq M$ .

Take  $\tau > 0$  with  $\tau \dim X \leq \theta$ . By Lemma 3.2 for every map  $\sigma : G \rightarrow \text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  and every  $\mathcal{W} \subseteq [d]$  with  $|\mathcal{W}| \geq (1 - \tau/2)d$  there exists  $\mathcal{C} \subseteq \mathcal{W}$  such that the map  $(s, c) \mapsto \sigma_s(c)$  from  $G \times \mathcal{C}$  to  $\sigma(G)\mathcal{C}$  is bijective and  $|\sigma(G)\mathcal{C}| \geq (1 - \tau)d$ .

Take  $\kappa > 0$  such that for any  $x, y \in X$  with  $\rho(x, y) \leq \kappa$  one has  $\rho(sx, sy) < \eta$  for all  $s \in G$ . Take  $\delta > 0$  with  $\delta^{1/2} < \kappa$  and  $\delta|\mathcal{U}||G| \leq \theta$ .

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . Now it suffices to show that if  $\sigma$  is a good enough sofic approximation and  $d$  is sufficiently large then

$$\frac{\mathcal{D}(\mathcal{U}, \rho, G, \delta, \sigma)}{d} \leq \lambda + 3\theta.$$

Denote by  $\mathcal{W}$  the subset of  $[d]$  consisting of  $a$  satisfying  $\sigma_s \sigma_{s^{-1}}(a) = \sigma_{e_G}(a) = a$  for all  $s \in G$ . Assuming that  $\sigma$  is a good enough sofic approximation, we have  $|\mathcal{W}| \geq (1 - \tau/2)d$  and can find  $\mathcal{C}$  as above. Set  $\mathcal{Z} = [d] \setminus \sigma(G)\mathcal{C}$ . Then  $|\mathcal{Z}| \leq \tau d$ .

For every  $\varphi \in \text{Map}(\rho, G, \delta, \sigma)$ , we have  $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) \leq \delta$  for all  $s \in G$ . Thus the set  $\Lambda_\varphi$  of all  $a \in [d]$  satisfying

$$\rho(\varphi(sa), s\varphi(a)) \leq \delta^{1/2}$$

for all  $s \in G$  has cardinality at least  $(1 - |G|\delta)d$ .

Take a partition of unity  $\{\zeta_U\}_{U \in \mathcal{U}}$  for  $X$  subordinate to  $\mathcal{U}$ . That is, each  $\zeta_U$  is a continuous function  $X \rightarrow [0, 1]$  with support contained in  $U$ , and

$$\sum_{U \in \mathcal{U}} \zeta_U = 1.$$

Define a continuous map  $\xi : X \rightarrow [0, 1]^{\mathcal{U}}$  by  $\xi(x)_U = \zeta_U(x)$  for  $x \in X$  and  $U \in \mathcal{U}$ . Consider the continuous map  $h : \text{Map}(\rho, G, \delta, \sigma) \rightarrow ([0, 1]^{\mathcal{U}})^{[d]}$  defined by

$$h(\varphi)_a = \xi(\varphi(a)) \max(\max_{s \in G} \rho(s\varphi(a), \varphi(sa)) - \kappa, 0)$$

for  $\varphi \in \text{Map}(\rho, G, \delta, \sigma)$  and  $a \in [d]$ . Denote by  $\nu$  the point of  $[0, 1]^{\mathcal{U}}$  having all coordinates 0. Set  $X_0$  to be the subset of  $([0, 1]^{\mathcal{U}})^{[d]}$  consisting of elements whose coordinates are equal to  $\nu$  at at least  $(1 - |G|\delta)d$  elements of  $[d]$ . For each  $\varphi \in \text{Map}(\rho, G, \delta, \sigma)$ , note that  $h(\varphi)_a = \nu$  for all  $a \in \Lambda_\varphi$  by our choice of  $\delta$  and hence  $h(\varphi) \in X_0$ . Thus we may think of  $h$  as a map from  $\text{Map}(\rho, G, \delta, \sigma)$  into  $X_0$ . Since the union of finitely many closed subsets of dimension at most  $m$  has dimension at most  $m$  [13, page 30], we get  $\dim X_0 \leq |\mathcal{U}||G|\delta d \leq \theta d$ .

Now define a continuous map  $\Psi : \text{Map}(\rho, G, \delta, \sigma) \rightarrow X_0 \times (\prod_{c \in \mathcal{C}} X) \times (\prod_{a \in \mathcal{Z}} X)$  as follows. For  $\varphi \in \text{Map}(\rho, G, \delta, \sigma)$ , the coordinate of  $\Psi(\varphi)$  in  $X_0$  is  $h(\varphi)$ , in  $X$  for  $c \in \mathcal{C}$  is  $\varphi(c)$ , in  $X$  for  $a \in \mathcal{Z}$  is  $\varphi(a)$ . We claim that  $\Psi$  is  $\mathcal{U}^d|_{\text{Map}(\rho, G, \delta, \sigma)}$ -compatible. Let  $w \in X_0 \times (\prod_{c \in \mathcal{C}} X) \times (\prod_{a \in \mathcal{Z}} X)$ . We need to show that for each  $a \in [d]$  there is some  $U \in \mathcal{U}$  depending only on  $w$  and  $a$  such that  $\varphi(a) \in U$  for every  $\varphi \in \Psi^{-1}(w)$ . We write the coordinates of  $w$  in  $X_0$ ,  $\prod_{c \in \mathcal{C}} X$ , and  $\prod_{a \in \mathcal{Z}} X$  as  $w^1$ ,  $w^2$ , and  $w^3$  respectively.

For each  $a \in \mathbb{Z}$ , one has  $w_a^3 \in U_{w_a^3}$  for some  $U_{w_a^3} \in \mathcal{U}$ . Then  $\varphi(a) \in U_{w_a^3}$  for every  $\varphi \in \Psi^{-1}(w)$  and  $a \in \mathbb{Z}$ .

For every  $a \in [d] \setminus \mathbb{Z}$ , we distinguish the two cases  $w_a^1 \neq \nu$  and  $w_a^1 = \nu$ . If  $w_a^1 \neq \nu$ , then  $(w_a^1)_U \neq 0$  for some  $U \in \mathcal{U}$ , and then for  $\varphi \in \Psi^{-1}(w)$  one has  $\zeta_U(\varphi(a)) > 0$  and hence  $\varphi(a) \in U$ . Suppose that  $w_a^1 = \nu$ . Say,  $a = \sigma(s^{-1})c$  for some  $s^{-1} \in G$  and  $c \in \mathcal{C}$ . Then  $\{x \in X : \rho(x, s^{-1}\varphi(c)) < \eta\} \subseteq U$  for some  $U \in \mathcal{U}$ . Let  $\varphi \in \Psi^{-1}(w)$ . Since  $c \in \mathcal{C} \subseteq \mathcal{W}$ , one has  $sa = \sigma_s \sigma_{s^{-1}}(c) = c$ . As  $\{\zeta_U\}_{U \in \mathcal{U}}$  is a partition of unity of  $X$ ,  $\xi(\varphi(a)) \neq \nu$ . But  $h(\varphi)_a = w_a^1 = \nu$ . Thus  $\max_{s' \in G} \rho(s'\varphi(a), \varphi(s'a)) \leq \kappa$ . In particular, one has  $\rho(s\varphi(a), \varphi(c)) = \rho(s\varphi(a), \varphi(sa)) \leq \kappa$ . From our choice of  $\kappa$ , one gets  $\rho(\varphi(a), s^{-1}\varphi(c)) < \eta$ . Thus  $\varphi(a) \in U$ . This proves the claim.

From Lemma 3.4 we get

$$\mathcal{D}(\mathcal{U}, \rho, G, \delta, \sigma) \leq \dim X_0 \times \left( \prod_{c \in \mathcal{C}} X \right) \times \left( \prod_{a \in \mathbb{Z}} X \right).$$

Taking  $d$  to be sufficiently large, we have  $|\mathcal{C}| \geq M$  and hence  $\dim X^{|\mathcal{C}|} \leq |\mathcal{C}||G|(\lambda + \theta)$ . Since the dimension of the product of two compact metrizable spaces is at most the sum of the dimensions of the factors [13, page 33 and Theorem V.8], we have

$$\begin{aligned} \dim X_0 \times \left( \prod_{c \in \mathcal{C}} X \right) \times \left( \prod_{a \in \mathbb{Z}} X \right) &\leq \dim X_0 + \dim X^{|\mathcal{C}|} + |\mathbb{Z}| \dim X \\ &\leq \theta d + |\mathcal{C}||G|(\lambda + \theta) + \tau d \dim X \\ &\leq \theta d + d(\lambda + \theta) + \theta d \\ &= d(\lambda + 3\theta). \end{aligned}$$

Therefore  $\mathcal{D}(\mathcal{U}, \rho, G, \delta, \sigma) \leq d(\lambda + 3\theta)$  as desired.  $\square$

#### 4. SOFIC MEAN METRIC DIMENSION

In this section we define the sofic mean metric dimension and establish some basic properties for it.

We start with recalling the definitions of the lower box dimension for a compact metric space and the mean metric dimension for actions of countable amenable groups.

For a pseudometric space  $(Y, \rho)$  and  $\varepsilon > 0$  we denote by  $N_\varepsilon(Y, \rho)$  the maximal cardinality of  $\varepsilon$ -separated subsets of  $Y$  with respect to  $\rho$ .

The *lower box dimension* of a compact metric space  $(Y, \rho)$  is defined as

$$\underline{\dim}_B(Y, \rho) := \liminf_{\varepsilon \rightarrow 0} \frac{\log N_\varepsilon(Y, \rho)}{|\log \varepsilon|}.$$

Let a countable (discrete) amenable group  $G$  act continuously on a compact metrizable space  $X$ . Let  $\rho$  be a continuous pseudometric on  $X$ . For a finite open

cover  $\mathcal{U}$  of  $X$ , we define the mesh of  $\mathcal{U}$  under  $\rho$  by

$$\text{mesh}(\mathcal{U}, \rho) = \max_{U \in \mathcal{U}} \text{diam}(U, \rho).$$

For a nonempty finite subset  $F$  of  $G$ , we define a pseudometric  $\rho_F$  on  $X$  by

$$\rho_F(x, y) = \max_{s \in F} \rho(sx, sy)$$

for  $x, y \in X$ . The function  $F \mapsto \log \min_{\text{mesh}(\mathcal{U}, \rho_F) < \varepsilon} |\mathcal{U}|$  defined on the set of nonempty finite subsets of  $G$  satisfies the conditions of the Ornstein-Weiss lemma [22] [20, Theorem 6.1], thus  $\min_{\text{mesh}(\mathcal{U}, \rho_F) < \varepsilon} \frac{\log |\mathcal{U}|}{|F|}$  converges to some real number, denoted by  $S(X, \varepsilon, \rho)$ , when  $F$  becomes more and more left invariant. The *mean metric dimension of  $X$  with respect to  $\rho$*  [20, page 13] is defined as

$$\text{mdim}_M(X, \rho) = \lim_{\varepsilon \rightarrow 0} \frac{S(X, \varepsilon, \rho)}{|\log \varepsilon|}.$$

As discussed on page 14 of [20], one can also write  $\text{mdim}_M(X, \rho)$  in a way similar to  $\underline{\dim}_B(Y, \rho)$ :

$$\text{mdim}_M(X, \rho) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \overline{\lim}_{n \rightarrow \infty} \frac{\log N_\varepsilon(X, \rho_{F_n})}{|F_n|},$$

for any left Følner sequence  $\{F_n\}_{n \in \mathbb{N}}$  of  $G$ , i.e. each  $F_n$  is a nonempty finite subset of  $G$  and  $\frac{|sF_n \setminus F_n|}{|F_n|} \rightarrow 0$  as  $n \rightarrow \infty$  for every  $s \in G$ . Also, define

$$\text{mdim}_M(X) = \inf_{\rho} \text{mdim}_M(X, \rho)$$

for  $\rho$  ranging over compatible metrics on  $X$ .

In the rest of this section we fix a countable sofic group  $G$  and a sofic approximation sequence  $\Sigma = \{\sigma_i : G \rightarrow \text{Sym}(d_i)\}_{i=1}^\infty$  for  $G$ . We also fix a continuous action  $\alpha$  of  $G$  on a compact metrizable space  $X$ .

As we discussed in Section 2, when defining sofic invariants, we replace  $X$  by  $\text{Map}(\rho, F, \delta, \sigma)$ . We also replace  $(\rho_F, \varepsilon)$ -separated subsets of  $X$  by  $(\rho_\infty, \varepsilon)$ -separated subsets of  $\text{Map}(\rho, F, \delta, \sigma)$ .

**Definition 4.1.** Let  $F$  be a nonempty finite subset of  $G$  and  $\delta > 0$ . For  $\varepsilon > 0$  and  $\rho$  a continuous pseudometric on  $X$  we define

$$\begin{aligned} h_{\Sigma, \infty}^\varepsilon(\rho, F, \delta) &= \overline{\lim}_{i \rightarrow \infty} \frac{1}{d_i} \log N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma_i), \rho_\infty), \\ h_{\Sigma, \infty}^\varepsilon(\rho, F) &= \inf_{\delta > 0} h_{\Sigma, \infty}^\varepsilon(\rho, F, \delta), \\ h_{\Sigma, \infty}^\varepsilon(\rho) &= \inf_F h_{\Sigma, \infty}^\varepsilon(\rho, F), \end{aligned}$$

where  $F$  in the third line ranges over the nonempty finite subsets of  $G$ . If  $\text{Map}(\rho, F, \delta, \sigma_i)$  is empty for all sufficiently large  $i$ , we set  $h_{\Sigma, \infty}^\varepsilon(\rho, F, \delta) = -\infty$ . We define the *sofic*

mean metric dimension of  $\alpha$  with respect to  $\rho$  as

$$\text{mdim}_{\Sigma, M}(X, \rho) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} h_{\Sigma, \infty}^{\varepsilon}(\rho).$$

We also define

$$\text{mdim}_{\Sigma, M}(X) = \inf_{\rho} \text{mdim}_{\Sigma, M}(X, \rho),$$

for  $\rho$  ranging over compatible metrics on  $X$ .

The *sofic topological entropy*  $h_{\Sigma}(X)$  of  $\alpha$  was defined in [14, Definition 4.6]. It was shown in Proposition 2.4 of [15] that

$$h_{\Sigma}(X) = \lim_{\varepsilon \rightarrow 0} h_{\Sigma, \infty}^{\varepsilon}(\rho)$$

for every compatible metric  $\rho$  on  $X$ . Thus we have

**Proposition 4.2.** *If  $h_{\Sigma}(X) < +\infty$ , then  $\text{mdim}_{\Sigma, M}(X, \rho) \leq 0$  for every compatible metric  $\rho$  on  $X$ .*

Then amenable group case of Proposition 4.2 was observed by Lindenstrauss and Weiss [20, page 14].

We say that a continuous pseudometric  $\rho$  on  $X$  is *dynamically generating* [18, Sect. 4] if for any distinct points  $x, y \in X$  one has  $\rho(sx, sy) > 0$  for some  $s \in G$ .

**Lemma 4.3.** *Let  $\rho$  be a dynamically generating continuous pseudometric on  $X$ . Enumerate the elements of  $G$  as  $s_1, s_2, \dots$ . Define  $\rho'$  by  $\rho'(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho(s_n x, s_n y)$  for all  $x, y \in X$ . Then  $\rho'$  is a compatible metric on  $X$ . Furthermore, if  $e_G = s_m$ , then for any  $\varepsilon > 0$  one has*

$$h_{\Sigma, \infty}^{4\varepsilon}(\rho') \leq h_{\Sigma, \infty}^{\varepsilon}(\rho) \leq h_{\Sigma, \infty}^{\varepsilon/2^m}(\rho').$$

In particular,  $\text{mdim}_{\Sigma, M}(X, \rho) = \text{mdim}_{\Sigma, M}(X, \rho')$ .

*Proof.* Clearly  $\rho'$  is a continuous pseudometric on  $X$ . Since  $\rho$  is dynamically generating,  $\rho'$  separates the points of  $X$ . Thus  $\rho'$  is a compatible metric on  $X$ . Let  $\varepsilon > 0$ .

We show first  $h_{\Sigma, \infty}^{\varepsilon}(\rho) \leq h_{\Sigma, \infty}^{\varepsilon/2^m}(\rho')$ . Let  $F$  be a finite subset of  $G$  containing  $e_G$  and  $\delta > 0$ . Take  $k \in \mathbb{N}$  with  $2^{-k} \text{diam}(X, \rho) < \delta/2$ . Set  $F' = \bigcup_{n=1}^k s_n F$  and take  $1 > \delta' > 0$  to be small which we shall fix in a moment.

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$ . We claim that  $\text{Map}(\rho, F', \delta', \sigma) \subseteq \text{Map}(\rho', F, \delta, \sigma)$ . Let  $\varphi \in \text{Map}(\rho, F', \delta', \sigma)$ . Then  $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) \leq \delta'$  for all  $s \in F'$ . Thus

$$|\{a \in [d] : \rho(\varphi \circ \sigma_s(a), \alpha_s \circ \varphi(a)) \leq \sqrt{\delta'}\}| \geq (1 - \delta')d$$

for every  $s \in F'$ , and hence

$$|\mathcal{W}| \geq (1 - \delta'|F'|)d,$$

for

$$\mathcal{W} := \{a \in [d] : \max_{s \in F'} \rho(\varphi \circ \sigma_s(a), \alpha_s \circ \varphi(a)) \leq \sqrt{\delta'}\}.$$

Set  $\mathcal{R} = \mathcal{W} \cap \bigcap_{t \in F} \sigma_t^{-1}(\mathcal{W})$ . Then  $|\mathcal{R}| \geq (1 - \delta'|F'|(1 + |F|))d$ . Also set

$$\mathcal{Q} = \{a \in [d] : \sigma_{s_n} \circ \sigma_t(a) = \sigma_{s_nt}(a) \text{ for all } 1 \leq n \leq k \text{ and } t \in F\}.$$

For any  $a \in \mathcal{R} \cap \mathcal{Q}$  and  $t \in F$ , since  $a, \sigma_t(a) \in \mathcal{W}$  and  $s_n, s_nt \in F'$  for all  $1 \leq n \leq k$ , we have

$$\begin{aligned} & \rho'(\varphi \circ \sigma_t(a), \alpha_t \circ \varphi(a)) \\ & \leq 2^{-k} \text{diam}(X, \rho) + \sum_{n=1}^k \frac{1}{2^n} \rho(\alpha_{s_n} \circ \varphi \circ \sigma_t(a), \alpha_{s_n} \circ \alpha_t \circ \varphi(a)) \\ & \leq \delta/2 + \sum_{n=1}^k \frac{1}{2^n} \left( \rho(\alpha_{s_n} \circ \varphi \circ \sigma_t(a), \varphi \circ \sigma_{s_n} \circ \sigma_t(a)) + \rho(\varphi \circ \sigma_{s_nt}(a), \alpha_{s_nt} \circ \varphi(a)) \right) \\ & \leq \delta/2 + \sum_{n=1}^k \frac{1}{2^n} \cdot 2\sqrt{\delta'} \leq \delta/2 + 2\sqrt{\delta'}. \end{aligned}$$

When  $\sigma$  is a good enough sofic approximation for  $G$ , one has  $|\mathcal{Q}| \geq (1 - \delta'|F'|)d$  and hence for any  $t \in F$ ,

$$\begin{aligned} (\rho'_2(\varphi \circ \sigma_t, \alpha_t \circ \varphi))^2 & \leq \frac{1}{d} (|\mathcal{R} \cap \mathcal{Q}|(\delta/2 + 2\sqrt{\delta'})^2 + (d - |\mathcal{R} \cap \mathcal{Q}|)(\text{diam}(X, \rho'))^2) \\ & \leq (\delta/2 + 2\sqrt{\delta'})^2 + \delta'|F'|(2 + |F|)(\text{diam}(X, \rho'))^2 < \delta^2, \end{aligned}$$

when  $\delta'$  is small enough independent of  $\sigma$  and  $\varphi$ . Therefore  $\varphi \in \text{Map}(\rho', F, \delta, \sigma)$ . This proves the claim.

Note that  $\frac{1}{2^m} \rho_\infty \leq \rho'_\infty$  on  $\text{Map}(\rho, F', \delta', \sigma)$ . Thus

$$N_\varepsilon(\text{Map}(\rho, F', \delta', \sigma), \rho_\infty) \leq N_{\varepsilon/2^m}(\text{Map}(\rho, F', \delta', \sigma), \rho'_\infty) \leq N_{\varepsilon/2^m}(\text{Map}(\rho', F, \delta, \sigma), \rho'_\infty),$$

when  $\sigma$  is a good enough sofic approximation for  $G$ . It follows that  $h_{\Sigma, \infty}^\varepsilon(\rho, F', \delta') \leq h_{\Sigma, \infty}^{\varepsilon/2^m}(\rho', F, \delta)$ , and hence  $h_{\Sigma, \infty}^\varepsilon(\rho) \leq h_{\Sigma, \infty}^{\varepsilon/2^m}(\rho')$  as desired.

Next we show  $h_{\Sigma, \infty}^{4\varepsilon}(\rho') \leq h_{\Sigma, \infty}^\varepsilon(\rho)$ . It suffices to show  $h_{\Sigma, \infty}^{4\varepsilon}(\rho') \leq h_{\Sigma, \infty}^\varepsilon(\rho) + \theta$  for every  $\theta > 0$ . Take  $k \in \mathbb{N}$  with  $2^{-k} \text{diam}(X, \rho) < \varepsilon/2$ .

Let  $F$  be a finite subset of  $G$  containing  $\{s_1, \dots, s_k\}$  and  $\delta > 0$  be sufficiently small which we shall specify in a moment. Set  $\delta' = \delta/2^m$ .

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some sufficiently large  $d \in \mathbb{N}$ .

Note that  $\frac{1}{2^m} \rho_2(\varphi, \psi) \leq \rho'_2(\varphi, \psi)$  for all maps  $\varphi, \psi : [d] \rightarrow X$ . Thus  $\text{Map}(\rho, F, \delta, \sigma) \supseteq \text{Map}(\rho', F, \delta', \sigma)$ .

Let  $\mathcal{E}$  be a  $(\rho'_\infty, 4\varepsilon)$ -separated subset of  $\text{Map}(\rho', F, \delta', \sigma)$  with  $|\mathcal{E}| = N_{4\varepsilon}(\text{Map}(\rho', F, \delta', \sigma), \rho'_\infty)$ . For each  $\varphi \in \mathcal{E}$  denote by  $\mathcal{W}_\varphi$  the set of  $a \in [d]$  satisfying  $\rho(\alpha_s \circ \varphi(a), \varphi \circ \sigma_s(a)) \leq \sqrt{\delta}$  for all  $s \in F$ . Then  $|\mathcal{W}_\varphi| \geq (1 - |F|\delta)d$ . Take  $\delta$  to be small enough so that  $|F|\delta < 1/2$ .



The number of subsets of  $[d]$  of cardinality at most  $|F|\delta d$  is equal to  $\sum_{j=0}^{\lfloor |F|\delta d \rfloor} \binom{d}{j}$ , which is at most  $|F|\delta d \binom{d}{\lfloor |F|\delta d \rfloor}$ , which by Stirling's approximation is less than  $\exp(\beta d)$  for some  $\beta > 0$  depending on  $\delta$  and  $|F|$  but not on  $d$  when  $d$  is sufficiently large with  $\beta \rightarrow 0$  as  $\delta \rightarrow 0$  for a fixed  $|F|$ . Take  $\delta$  to be small enough such that  $\beta < \theta/2$ . Then, when  $d$  is sufficiently large, we can find a subset  $\mathcal{F}$  of  $\mathcal{E}$  with  $|\mathcal{F}| \exp(\beta d) \geq |\mathcal{E}|$  such that  $\mathcal{W}_\varphi$  is the same, say  $\mathcal{W}$ , for every  $\varphi \in \mathcal{F}$ .

Let  $\varphi \in \mathcal{F}$ . Let us estimate how many elements in  $\mathcal{F}$  are in the open ball  $B_\varphi := \{\psi \in X^{[d]} : \rho_\infty(\varphi, \psi) < \varepsilon\}$ . Let  $\psi \in \mathcal{F} \cap B_\varphi$ . For any  $a \in \mathcal{W}$  and  $s \in F$ , we have

$$\begin{aligned} \rho(s\varphi(a), s\psi(a)) &\leq \rho(s\varphi(a), \varphi(sa)) + \rho(\varphi(sa), \psi(sa)) + \rho(\psi(sa), s\psi(a)) \\ &\leq \sqrt{\delta} + \varepsilon + \sqrt{\delta} \leq \frac{3}{2}\varepsilon, \end{aligned}$$

when  $\delta$  is taken to be small enough. It follows that for any  $a \in \mathcal{W}$  we have

$$\begin{aligned} \rho'(\varphi(a), \psi(a)) &\leq 2^{-k} \text{diam}(X, \rho) + \sum_{n=1}^k 2^{-n} \rho(s_n \varphi(a), s_n \psi(a)) \\ &< \frac{1}{2}\varepsilon + \frac{3}{2}\varepsilon = 2\varepsilon. \end{aligned}$$

Then  $\rho'_\infty(\varphi|_{\mathcal{W}}, \psi|_{\mathcal{W}}) < 2\varepsilon$ .

Let  $Y$  be a maximal  $(\rho', 2\varepsilon)$ -separated subset of  $X$ . Set  $\mathcal{W}^c = [d] \setminus \mathcal{W}$ . For each  $\psi \in \mathcal{F} \cap B_\varphi$ , there exists some  $f_\psi \in Y^{\mathcal{W}^c}$  with  $\rho'_\infty(\psi|_{\mathcal{W}^c}, f_\psi) < 2\varepsilon$ . Then we can find a subset  $\mathcal{A}$  of  $\mathcal{F} \cap B_\varphi$  with  $|Y|^{\mathcal{W}^c} |\mathcal{A}| \geq |\mathcal{F} \cap B_\varphi|$  such that  $f_\psi$  is the same, say  $f$ , for every  $\psi \in \mathcal{A}$ . For any  $\psi, \psi' \in \mathcal{A}$ , we have

$$\rho'_\infty(\psi|_{\mathcal{W}^c}, \psi'|_{\mathcal{W}^c}) \leq \rho'_\infty(\psi|_{\mathcal{W}^c}, f) + \rho'_\infty(f, \psi'|_{\mathcal{W}^c}) < 4\varepsilon,$$

and

$$\rho'_\infty(\psi|_{\mathcal{W}}, \psi'|_{\mathcal{W}}) \leq \rho'_\infty(\psi|_{\mathcal{W}}, \varphi|_{\mathcal{W}}) + \rho'_\infty(\varphi|_{\mathcal{W}}, \psi'|_{\mathcal{W}}) < 4\varepsilon,$$

and hence  $\rho'_\infty(\psi, \psi') < 4\varepsilon$ . Since  $\mathcal{A}$  is  $(\rho'_\infty, 4\varepsilon)$ -separated, we get  $\psi = \psi'$ . Therefore  $|\mathcal{A}| \leq 1$ , and hence

$$|\mathcal{F} \cap B_\varphi| \leq |Y|^{\mathcal{W}^c} |\mathcal{A}| \leq |Y|^{|F|\delta d}.$$

Let  $\mathcal{B}$  be a maximal  $(\rho_\infty, \varepsilon)$ -separated subset of  $\mathcal{F}$ . Then  $\mathcal{F} = \bigcup_{\varphi \in \mathcal{B}} (\mathcal{F} \cap B_\varphi)$ . Thus

$$\begin{aligned} N_{4\varepsilon}(\text{Map}(\rho', F, \delta', \sigma), \rho'_\infty) &= |\mathcal{E}| \leq \exp(\beta d) |\mathcal{F}| \leq \exp(\theta d/2) |\mathcal{B}| |Y|^{|F|\delta d} \\ &\leq \exp(\theta d/2) |Y|^{|F|\delta d} N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma), \rho_\infty) \\ &\leq \exp(\theta d) N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma), \rho_\infty), \end{aligned}$$

when we take  $\delta$  to be small enough. Therefore  $h_{\Sigma, \infty}^{4\varepsilon}(\rho', F, \delta') \leq h_{\Sigma, \infty}^\varepsilon(\rho, F, \delta) + \theta$ . It follows that  $h_{\Sigma, \infty}^{4\varepsilon}(\rho') \leq h_{\Sigma, \infty}^\varepsilon(\rho) + \theta$  as desired.  $\square$

From Lemma 4.3 we get

**Proposition 4.4.** *One has*

$$\text{mdim}_{\Sigma, \mathbf{M}}(X) = \inf_{\rho} \text{mdim}_{\Sigma, \mathbf{M}}(X, \rho)$$

for  $\rho$  ranging over dynamically generating continuous pseudometrics on  $X$ .

The following is the analogue of Proposition 2.5.

**Proposition 4.5.** *Let  $G$  act continuously on a compact metrizable space  $X$ . Let  $Y$  be a closed  $G$ -invariant subset of  $X$ . Then  $\text{mdim}_{\Sigma, \mathbf{M}}(Y) \leq \text{mdim}_{\Sigma, \mathbf{M}}(X)$ .*

*Proof.* Let  $\rho$  be a compatible metric on  $X$ . Then  $\rho$  restricts to a compatible metric  $\rho'$  on  $Y$ .

Note that  $\text{Map}(\rho', F, \delta, \sigma) \subseteq \text{Map}(\rho, F, \delta, \sigma)$  for any nonempty finite subset  $F$  of  $G$ , any  $\delta > 0$ , and any map  $\sigma$  from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . Furthermore, the restriction of  $\rho_{\infty}$  on  $\text{Map}(\rho', F, \delta, \sigma)$  is exactly  $\rho'_{\infty}$ . Thus  $N_{\varepsilon}(\text{Map}(\rho', F, \delta, \sigma), \rho'_{\infty}) \leq N_{\varepsilon}(\text{Map}(\rho, F, \delta, \sigma), \rho_{\infty})$  for any  $\varepsilon > 0$ . It follows that  $\text{mdim}_{\Sigma, \mathbf{M}}(Y) \leq \text{mdim}_{\Sigma, \mathbf{M}}(Y, \rho') \leq \text{mdim}_{\Sigma, \mathbf{M}}(X, \rho)$ . Since  $\rho$  is an arbitrary compatible metric on  $X$ , we get  $\text{mdim}_{\Sigma, \mathbf{M}}(Y) \leq \text{mdim}_{\Sigma, \mathbf{M}}(X)$ .  $\square$

## 5. SOFIC MEAN METRIC DIMENSION FOR AMENABLE GROUPS

In this section we show that the sofic mean metric dimension extends the mean metric dimension for actions of countable amenable groups:

**Theorem 5.1.** *Let a countable (discrete) amenable group  $G$  act continuously on a compact metrizable space  $X$ . Let  $\Sigma$  be a sofic approximation sequence for  $G$ . Then*

$$\text{mdim}_{\Sigma, \mathbf{M}}(X, \rho) = \text{mdim}_{\mathbf{M}}(X, \rho)$$

for every continuous pseudometric  $\rho$  on  $X$ . In particular,

$$\text{mdim}_{\Sigma, \mathbf{M}}(X) = \text{mdim}_{\mathbf{M}}(X).$$

Theorem 5.1 follows directly from Lemmas 5.2 and 5.3 below.

**Lemma 5.2.** *Let a countable amenable group  $G$  act continuously on a compact metrizable space  $X$ . Let  $\rho$  be a continuous pseudometric on  $X$ . Then for any  $\varepsilon > 0$  we have  $h_{\Sigma, \infty}^{\varepsilon}(\rho) \geq S(X, 2\varepsilon, \rho)$ . In particular,  $\text{mdim}_{\Sigma, \mathbf{M}}(X, \rho) \geq \text{mdim}_{\mathbf{M}}(X, \rho)$ .*

*Proof.* It suffices to show that  $h_{\Sigma, \infty}^{\varepsilon}(\rho) \geq S(X, 2\varepsilon, \rho) - 2\theta$  for every  $\theta > 0$ .

Take a nonempty finite subset  $K$  of  $G$  and  $\varepsilon' > 0$  such that for any nonempty finite subset  $F'$  of  $G$  satisfying  $|KF' \setminus F'| < \varepsilon'|F'|$ , one has

$$\frac{1}{|F'|} \log N_{\varepsilon}(X, \rho_{F'}) \geq \frac{1}{|F'|} \log \min_{\text{mesh}(\mathcal{U}, \rho_{F'}) < 2\varepsilon} |\mathcal{U}| \geq S(X, 2\varepsilon, \rho) - \theta.$$

Let  $F$  be a nonempty finite subset of  $G$  and  $\delta > 0$ . Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . Now it suffices to show that if  $\sigma$  is a good enough sofic approximation then

$$\frac{1}{d} \log N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma), \rho_\infty) \geq S(X, 2\varepsilon, \rho) - 2\theta.$$

Take  $\delta' > 0$  such that  $\sqrt{\delta'} \text{diam}(X, \rho) < \delta/2$  and  $(1 - \delta')(S(X, 2\varepsilon, \rho) - \theta) \geq S(X, 2\varepsilon, \rho) - 2\theta$ . By Lemma 3.2 there are an  $\ell \in \mathbb{N}$  and nonempty finite subsets  $F_1, \dots, F_\ell$  of  $G$  satisfying  $|KF_k \setminus F_k| < \min(\varepsilon', \delta')|F_k|$  for all  $k = 1, \dots, \ell$  such that for every map  $\sigma : G \rightarrow \text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  and every  $\mathcal{W} \subseteq [d]$  with  $|\mathcal{W}| \geq (1 - \delta'/2)d$  there exist  $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$  satisfying the following:

- (1) for every  $k = 1, \dots, \ell$ , the map  $(s, c) \mapsto \sigma_s(c)$  from  $F_k \times \mathcal{C}_k$  to  $\sigma(F_k)\mathcal{C}_k$  is bijective,
- (2) the sets  $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_\ell)\mathcal{C}_\ell$  are pairwise disjoint and  $|\bigcup_{k=1}^\ell \sigma(F_k)\mathcal{C}_k| \geq (1 - \delta')d$ .

Let  $\sigma : G \rightarrow \text{Sym}(d)$  for some  $d \in \mathbb{N}$  be a good enough sofic approximation for  $G$  such that  $|\mathcal{W}| \geq (1 - \delta'/2)d$  for

$$\mathcal{W} := \{a \in [d] : \sigma_t \sigma_s(a) = \sigma_{ts}(a) \text{ for all } t \in F, s \in \bigcup_{k=1}^\ell F_k\}.$$

Then we have  $\mathcal{C}_1, \dots, \mathcal{C}_\ell$  as above.

For every  $k \in \{1, \dots, \ell\}$  pick an  $\varepsilon$ -separated set  $E_k \subseteq X$  with respect to  $\rho_{F_k}$  of maximal cardinality. Then

$$\frac{1}{|F_k|} \log |E_k| = \frac{1}{|F_k|} \log N_\varepsilon(X, \rho_{F_k}) \geq S(X, 2\varepsilon, \rho) - \theta.$$

For each  $h = (h_k)_{k=1}^\ell \in \prod_{k=1}^\ell (E_k)^{\mathcal{C}_k}$  take a map  $\varphi_h : [d] \rightarrow X$  such that

$$\varphi_h(sc) = s(h_k(c))$$

for all  $k \in \{1, \dots, \ell\}$ ,  $c \in \mathcal{C}_k$ , and  $s \in F_k$ . Note that if  $t \in F$ ,  $k \in \{1, \dots, \ell\}$ ,  $s \in F_k$ ,  $c \in \mathcal{C}_k$ , and  $ts \in F_k$ , then  $\sigma_t \sigma_s(c) = \sigma_{ts}(c)$ , and hence  $\alpha_t \circ \varphi_h(sc) = \varphi_h \circ \sigma_t(sc)$ . It follows that  $\rho_2(\alpha_t \circ \varphi_h, \varphi_h \circ \sigma_t) < \delta$  for all  $t \in F$ , and thus  $\varphi_h \in \text{Map}(\rho, F, \delta, \sigma)$ .

Now if  $h = (h_k)_{k=1}^\ell$  and  $h' = (h'_k)_{k=1}^\ell$  are distinct elements of  $\prod_{k=1}^\ell (E_k)^{\mathcal{C}_k}$ , then  $h_k(c) \neq h'_k(c)$  for some  $k \in \{1, \dots, \ell\}$  and  $c \in \mathcal{C}_k$ . Since  $h_k(c)$  and  $h'_k(c)$  are distinct points in  $E_k$  which is  $\varepsilon$ -separated with respect to  $\rho_{F_k}$ ,  $h_k(c)$  and  $h'_k(c)$  are  $\varepsilon$ -separated with respect to  $\rho_{F_k}$ , and thus we have  $\rho_\infty(\varphi_h, \varphi_{h'}) \geq \varepsilon$ . Therefore

$$\frac{1}{d} \log N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma), \rho_\infty) \geq \frac{1}{d} \sum_{k=1}^\ell |\mathcal{C}_k| \log |E_k|$$

$$\begin{aligned}
&\geq \frac{1}{d} \sum_{k=1}^{\ell} |\mathcal{C}_k| |F_k| (S(X, 2\varepsilon, \rho) - \theta) \\
&\geq (1 - \delta') (S(X, 2\varepsilon, \rho) - \theta) \\
&\geq S(X, 2\varepsilon, \rho) - 2\theta,
\end{aligned}$$

as desired.  $\square$

**Lemma 5.3.** *Let a countable amenable group  $G$  act continuously on a compact metrizable space  $X$ . Let  $\rho$  be a continuous pseudometric on  $X$ . Then for any  $\varepsilon > 0$  we have  $h_{\Sigma, \infty}^{\varepsilon}(\rho) \leq S(X, \varepsilon/4, \rho)$ . In particular,  $\text{mdim}_{\Sigma, M}(X, \rho) \leq \text{mdim}_M(X, \rho)$ .*

*Proof.* Let  $\varepsilon > 0$ . It suffices to show that  $h_{\Sigma, \infty}^{\varepsilon}(\rho) \leq S(X, \varepsilon/4, \rho) + 3\theta$  for every  $\theta > 0$ .

Take a nonempty finite subset  $K$  of  $G$  and  $\delta' > 0$  such that  $\min_{\text{mesh}(\mathcal{U}, \rho_{F'}) < \varepsilon/4} |\mathcal{U}| < \exp((S(X, \varepsilon/4, \rho) + \theta)|F'|)$  for every nonempty finite subset  $F'$  of  $G$  satisfying  $|KF' \setminus F'| < \delta'|F'|$ . Take an  $\eta \in (0, 1)$  such that  $(N_{\varepsilon/4}(X, \rho))^{2\eta} \leq \exp(\theta)$ .

By Lemma 3.2 there are an  $\ell \in \mathbb{N}$  and nonempty finite subsets  $F_1, \dots, F_{\ell}$  of  $G$  satisfying  $|KF_k \setminus F_k| < \delta'|F_k|$  for all  $k = 1, \dots, \ell$  such that for every map  $\sigma : G \rightarrow \text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  and every  $\mathcal{W} \subseteq [d]$  with  $|\mathcal{W}| \geq (1 - \eta)d$  there exist  $\mathcal{C}_1, \dots, \mathcal{C}_{\ell} \subseteq \mathcal{W}$  satisfying the following:

- (1) for every  $k = 1, \dots, \ell$ , the map  $(s, c) \mapsto \sigma_s(c)$  from  $F_k \times \mathcal{C}_k$  to  $\sigma(F_k)\mathcal{C}_k$  is bijective,
- (2) the sets  $\sigma(F_1)\mathcal{C}_1, \dots, \sigma(F_{\ell})\mathcal{C}_{\ell}$  are pairwise disjoint and  $|\bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k| \geq (1 - 2\eta)d$ .

Then

$$(1) \quad N_{\varepsilon/4}(X, \rho_{F_k}) \leq \min_{\text{mesh}(\mathcal{U}, \rho_{F_k}) < \varepsilon/4} |\mathcal{U}| < \exp((S(X, \varepsilon/4, \rho) + \theta)|F_k|)$$

for every  $k = 1, \dots, \ell$ .

Set  $F = \bigcup_{k=1}^{\ell} F_k$ . Let  $\delta > 0$  be a small positive number which we will determine in a moment. Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some sufficiently large  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$ . We will show that  $N_{\varepsilon}(\text{Map}(\rho, F, \delta, \sigma), \rho_{\infty}) \leq \exp(S(X, \varepsilon/4, \rho) + 3\theta)d$ , which will complete the proof since we can then conclude that  $h_{\Sigma, \infty}^{\varepsilon}(\rho, F, \delta) \leq S(X, \varepsilon/4, \rho) + 3\theta$  and hence  $h_{\Sigma, \infty}^{\varepsilon}(\rho) \leq S(X, \varepsilon/4, \rho) + 3\theta$ .

For every  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$ , we have  $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) \leq \delta$  for all  $s \in F$ . Thus the set  $\mathcal{W}_{\varphi}$  of all  $a \in [d]$  satisfying

$$\rho(\varphi(sa), s\varphi(a)) \leq \sqrt{\delta}$$

for all  $s \in F$  has cardinality at least  $(1 - |F|\delta)d$ .

For each  $\mathcal{W} \subseteq [d]$  we define on the set of maps from  $[d]$  to  $X$  the pseudometric

$$\rho_{\mathcal{W}, \infty}(\varphi, \psi) = \rho_{\infty}(\varphi|_{\mathcal{W}}, \psi|_{\mathcal{W}}).$$

Take a  $(\rho_\infty, \varepsilon)$ -separated subset  $\mathcal{E}$  of  $\text{Map}(\rho, F, \delta, \sigma)$  of maximal cardinality.

Set  $n = |F|$ . When  $n\delta < 1/2$ , the number of subsets of  $[d]$  of cardinality no greater than  $n\delta d$  is equal to  $\sum_{j=0}^{\lfloor n\delta d \rfloor} \binom{d}{j}$ , which is at most  $n\delta d \binom{d}{n\delta d}$ , which by Stirling's approximation is less than  $\exp(\beta d)$  for some  $\beta > 0$  depending on  $\delta$  and  $n$  but not on  $d$  when  $d$  is sufficiently large with  $\beta \rightarrow 0$  as  $\delta \rightarrow 0$  for a fixed  $n$ . Thus when  $\delta$  is small enough and  $d$  is large enough, there is a subset  $\mathcal{F}$  of  $\mathcal{E}$  with  $\exp(\theta d)|\mathcal{F}| \geq |\mathcal{E}|$  such that the set  $\mathcal{W}_\varphi$  is the same, say  $\mathcal{W}$ , for every  $\varphi \in \mathcal{F}$ , and  $|\mathcal{W}|/d > 1 - \eta$ . Then we have  $\mathcal{C}_1, \dots, \mathcal{C}_\ell \subseteq \mathcal{W}$  as above.

Let  $1 \leq k \leq \ell$  and  $c \in \mathcal{C}_k$ . Take an  $(\varepsilon/2)$ -spanning subset  $\mathcal{D}_{k,c}$  of  $\mathcal{F}$  with respect to  $\rho_{\sigma(F_k)c, \infty}$  of minimal cardinality. We will show that  $|\mathcal{D}_{k,c}| \leq \exp((S(X, \varepsilon/4, \rho) + \theta)|F_k|)$  when  $\delta$  is small enough. To this end, let  $\mathcal{D}$  be an  $(\varepsilon/2)$ -separated subset of  $\mathcal{F}$  with respect to  $\rho_{\sigma(F_k)c, \infty}$ . For any two distinct elements  $\varphi$  and  $\psi$  of  $\mathcal{D}$  we have, for every  $s \in F_k$ , since  $c \in \mathcal{W}_\varphi \cap \mathcal{W}_\psi$ ,

$$\begin{aligned} \rho(s\varphi(c), s\psi(c)) &\geq \rho(\varphi(sc), \psi(sc)) - \rho(s\varphi(c), \varphi(sc)) - \rho(s\psi(c), \psi(sc)) \\ &\geq \rho(\varphi(sc), \psi(sc)) - 2\sqrt{\delta}, \end{aligned}$$

and hence

$$\rho_{F_k}(\varphi(c), \psi(c)) = \max_{s \in F_k} \rho(s\varphi(c), s\psi(c)) \geq \max_{s \in F_k} \rho(\varphi(sc), \psi(sc)) - 2\sqrt{\delta} > \varepsilon/2 - \varepsilon/4 = \varepsilon/4,$$

granted that  $\delta$  is taken small enough. Thus  $\{\varphi(c) : \varphi \in \mathcal{D}\}$  is a  $(\rho_{F_k}, \varepsilon/4)$ -separated subset of  $X$  of cardinality  $|\mathcal{D}|$ , so that

$$|\mathcal{D}| \leq N_{\varepsilon/4}(X, \rho_{F_k}) \stackrel{(1)}{\leq} \exp((S(X, \varepsilon/4, \rho) + \theta)|F_k|).$$

Therefore

$$|\mathcal{D}_{k,c}| \leq N_{\varepsilon/2}(\mathcal{F}, \rho_{\sigma(F_k)c, \infty}) \leq \exp((S(X, \varepsilon/4, \rho) + \theta)|F_k|),$$

as we wished to show.

Set

$$\mathcal{Z} = [d] \setminus \bigcup_{k=1}^{\ell} \sigma(F_k)\mathcal{C}_k,$$

and take an  $(\varepsilon/2)$ -spanning subset  $\mathcal{D}_{\mathcal{Z}}$  of  $\mathcal{F}$  with respect to  $\rho_{\mathcal{Z}, \infty}$  of minimal cardinality. We have

$$|\mathcal{D}_{\mathcal{Z}}| \leq (N_{\varepsilon/4}(X, \rho))^{|\mathcal{Z}|} \leq (N_{\varepsilon/4}(X, \rho))^{2\eta d}.$$

Write  $\mathcal{A}$  for the set of all maps  $\varphi : [d] \rightarrow X$  such that  $\varphi|_{\mathcal{Z}} \in \mathcal{D}_{\mathcal{Z}}|_{\mathcal{Z}}$  and  $\varphi|_{\sigma(F_k)c} \in \mathcal{D}_{k,c}|_{\sigma(F_k)c}$  for all  $1 \leq k \leq \ell$  and  $c \in \mathcal{C}_k$ . Then, by our choice of  $\eta$ ,

$$|\mathcal{A}| = |\mathcal{D}_{\mathcal{Z}}| \prod_{k=1}^{\ell} \prod_{c \in \mathcal{C}_k} |\mathcal{D}_{k,c}| \leq (N_{\varepsilon/4}(X, \rho))^{2\eta d} \exp\left(\sum_{k=1}^{\ell} \sum_{c \in \mathcal{C}_k} (S(X, \varepsilon/4, \rho) + \theta)|F_k|\right)$$

$$\begin{aligned}
&= (N_{\varepsilon/4}(X, \rho))^{2\eta d} \exp \left( (S(X, \varepsilon/4, \rho) + \theta) \sum_{k=1}^{\ell} |F_k| |C_k| \right) \\
&\leq \exp(\theta d) \exp((S(X, \varepsilon/4, \rho) + \theta)d) = \exp((S(X, \varepsilon/4, \rho) + 2\theta)d).
\end{aligned}$$

Now since every element of  $\mathcal{F}$  lies within  $\rho_\infty$ -distance  $\varepsilon/2$  to an element of  $\mathcal{A}$  and  $\mathcal{F}$  is  $\varepsilon$ -separated with respect to  $\rho_\infty$ , the cardinality of  $\mathcal{F}$  is at most that of  $\mathcal{A}$ . Therefore

$$\begin{aligned}
N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma), \rho_\infty) &= |\mathcal{E}| \leq \exp(\theta d) |\mathcal{F}| \leq \exp(\theta d) |\mathcal{A}| \\
&\leq \exp(\theta d) \exp((S(X, \varepsilon/4, \rho) + 2\theta)d) \\
&= \exp((S(X, \varepsilon/4, \rho) + 3\theta)d),
\end{aligned}$$

as desired.  $\square$

## 6. COMPARISON OF SOFIC MEAN DIMENSIONS

In this section we prove the following relation between the sofic mean dimensions:

**Theorem 6.1.** *Let a countable sofic group  $G$  act continuously on a compact metrizable space  $X$ . Let  $\Sigma$  be a sofic approximation sequence of  $G$ . Then*

$$\text{mdim}_\Sigma(X) \leq \text{mdim}_{\Sigma, M}(X).$$

The amenable group case of Theorem 6.1 was proved by Lindenstrauss and Weiss [20, Theorem 4.2]. We adapt their proof to our situation.

**Lemma 6.2.** *Let  $\mathcal{U}$  be a finite open cover of  $X$  and  $\rho$  be a compatible metric on  $X$ . Then there exists a Lipschitz function  $f_U : X \rightarrow [0, 1]$  vanishing on  $X \setminus U$  for each  $U \in \mathcal{U}$  such that  $\max_{U \in \mathcal{U}} f_U(x) = 1$  for every  $x \in X$ .*

*Proof.* Note that for each  $U \in \mathcal{U}$ , the function  $\rho(\cdot, X \setminus U) : X \rightarrow [0, +\infty)$  is Lipschitz and vanishes on  $X \setminus U$ . Furthermore,  $\sum_{U \in \mathcal{U}} \rho(x, X \setminus U) > 0$  for every  $x \in X$ . Define  $g_U, f_U : X \rightarrow [0, 1]$  by

$$g_U(x) = \frac{\rho(x, X \setminus U)}{\sum_{V \in \mathcal{U}} \rho(x, X \setminus V)},$$

and

$$f_U(x) = |\mathcal{U}| \min(g_U(x), 1/|\mathcal{U}|).$$

Then  $g_U$  is Lipschitz and vanishes on  $X \setminus U$ , and hence so is  $f_U$ . Furthermore, for each  $x \in X$ , one has  $\sum_{U \in \mathcal{U}} g_U(x) = 1$  and thus  $g_U(x) \geq 1/|\mathcal{U}|$  for some  $U \in \mathcal{U}$ . It follows that  $f_U(x) = 1$  for some  $U \in \mathcal{U}$ .  $\square$

Let  $\rho, \mathcal{U}$  and  $f_U$  be as in Lemma 6.2. Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . We define a continuous map  $\Phi_d : X^{[d]} \rightarrow [0, 1]^{[d] \times \mathcal{U}}$  by  $\Phi_d(\varphi)_{a,U} = f_U(\varphi(a))$  for  $a \in [d]$  and  $U \in \mathcal{U}$ .

**Lemma 6.3.** *Let  $\theta > 0$ , and set  $D = \text{mdim}_{\Sigma, M}(X, \rho)$ . Then there exist a nonempty finite subset  $F$  of  $G$ ,  $\delta > 0$  and  $M > 0$  such that for any  $i \in \mathbb{N}$  with  $i \geq M$ , there exists  $\xi \in (0, 1)^{[d_i] \times \mathcal{U}}$  such that for any  $S \subseteq [d_i] \times \mathcal{U}$  with  $|S| \geq (D + \theta)d_i$ , one has*

$$\xi|_S \notin \Phi_{d_i}(\text{Map}(\rho, F, \delta, \sigma_i))|_S.$$

*Proof.* Denote by  $C$  the supremum of the Lipschitz constants of  $f_U$  for all  $U \in \mathcal{U}$ . Then

$$\|\Phi_d(\varphi) - \Phi_d(\psi)\|_\infty \leq C\rho_\infty(\varphi, \psi)$$

for all  $d \in \mathbb{N}$  and  $\varphi, \psi \in X^{[d]}$ .

Take  $0 < \varepsilon < \min(1, (2C)^{-1})$  small enough, which we shall determine in a moment, satisfying

$$\frac{h_{\Sigma, \infty}^\varepsilon(\rho)}{|\log \varepsilon|} < D + \theta/3.$$

Then we can find a nonempty finite subset  $F$  of  $G$ ,  $\delta > 0$ , and  $M > 0$  such that for every  $i \in \mathbb{N}$  with  $i > M$ , we have

$$\frac{1}{d_i |\log \varepsilon|} \log N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma_i), \rho_\infty) \leq D + \theta/2,$$

that is,

$$N_\varepsilon(\text{Map}(\rho, F, \delta, \sigma_i), \rho_\infty) \leq \varepsilon^{-(D+\theta/2)d_i}.$$

It follows that we can cover  $\Phi_{d_i}(\text{Map}(\rho, F, \delta, \sigma_i))$  using  $\varepsilon^{-(D+\theta/2)d_i}$  balls in the  $\|\cdot\|_\infty$  norm with radius  $\varepsilon C$ .

Denote by  $\mu$  the Lebesgue measure on  $[0, 1]^{[d_i] \times \mathcal{U}}$ . For each  $S \subseteq [d_i] \times \mathcal{U}$ , the set  $\Phi_{d_i}(\text{Map}(\rho, F, \delta, \sigma_i))|_S \subseteq [0, 1]^S$  can be covered using  $\varepsilon^{-(D+\theta/2)d_i}$  balls in the  $\|\cdot\|_\infty$  norm with radius  $\varepsilon C$ , and hence has Lebesgue measure at most  $\varepsilon^{-(D+\theta/2)d_i} (2\varepsilon C)^{|S|}$ . Thus,

$$\mu(\{\xi \in [0, 1]^{[d_i] \times \mathcal{U}} : \xi|_S \in \Phi_{d_i}(\text{Map}(\rho, F, \delta, \sigma_i))|_S\}) \leq \varepsilon^{-(D+\theta/2)d_i} (2\varepsilon C)^{|S|}.$$

Therefore, the set of  $\xi \in [0, 1]^{[d_i] \times \mathcal{U}}$  satisfying  $\xi|_S \in \Phi_{d_i}(\text{Map}(\rho, F, \delta, \sigma_i))|_S$  for some  $S \subseteq [d_i] \times \mathcal{U}$  with  $|S| \geq (D + \theta)d_i$  has Lebesgue measure at most

$$2^{[d_i] \times \mathcal{U}} \varepsilon^{-(D+\theta/2)d_i} (2\varepsilon C)^{(D+\theta)d_i} = (2^{|\mathcal{U}|} \varepsilon^{\theta/2} C^{D+\theta})^{d_i} < 1,$$

when we take  $\varepsilon$  to be small enough. Thus we can find  $\xi$  with desired property.  $\square$

**Lemma 6.4.** *Let  $F$  be a nonempty finite subset of  $G$  and  $\delta > 0$ . Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . Let  $\Psi$  be a continuous map from  $\Phi_d(\text{Map}(\rho, F, \delta, \sigma))$  to  $[0, 1]^{[d] \times \mathcal{U}}$ . Suppose that for any  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$  and any  $(a, U) \in [d] \times \mathcal{U}$ , if  $\Phi_d(\varphi)_{a, U} = 0$  or 1, then  $\Psi(\Phi_d(\varphi))_{a, U} = 0$  or 1 accordingly. Then  $\Psi \circ \Phi_d$  is  $\mathcal{U}^d|_{\text{Map}(\rho, F, \delta, \sigma)}$ -compatible.*

*Proof.* Let  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$  and  $a \in [d]$ . By the choice of  $\{f_U\}_{U \in \mathcal{U}}$  there exists some  $U \in \mathcal{U}$  such that  $\Phi_d(\varphi)_{a,U} = f_U(\varphi(a)) = 1$ . By our assumption on  $\Psi$  we then have  $\Psi(\Phi_d(\varphi))_{a,U} = 1$ .

Let  $\zeta \in \Psi(\Phi_d(\text{Map}(\rho, F, \delta, \sigma)))$  and  $a \in [d]$ . It suffices to show that there exists some  $U \in \mathcal{U}$  such that one has  $\varphi(a) \in U$  for every  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$  satisfying  $\Psi(\Phi_d(\varphi)) = \zeta$ . By the above paragraph we can find some  $U \in \mathcal{U}$  such that  $\zeta_{a,U} = 1$ . Let  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$  with  $\Psi(\Phi_d(\varphi)) = \zeta$ . By our assumption on  $\Psi$  we have  $f_U(\varphi(a)) = \Phi_d(\varphi)_{a,U} > 0$ . Since  $f_U$  vanishes on  $X \setminus U$ , we get  $\varphi(a) \in U$ .  $\square$

**Lemma 6.5.** *Let  $W$  be a finite set and  $Z$  a closed subset of  $[0, 1]^W$ . Let  $m \in \mathbb{N}$  and  $\xi \in (0, 1)^W$  such that for every  $S \subseteq W$  with  $|S| \geq m$  one has  $\xi|_S \notin Z|_S$ . Then there exists a continuous map  $\Psi$  from  $Z$  into  $[0, 1]^W$  such that  $\dim \Psi(Z) \leq m$  and for any  $z \in Z$  and any  $w \in W$ , if  $z_w = 0$  or  $1$ , then  $\Psi(z)_w = 0$  or  $1$  accordingly.*

*Proof.* This can be proved as in the proof of [20, Theorem 4.2]. We give a slightly different proof.

For each  $S \subseteq W$ , denote by  $Y_S$  the subset of  $[0, 1]^W$  consisting of elements whose coordinate at any  $w \in W \setminus S$  is either 0 or 1. Then  $Y_S$  is a closed subset of  $[0, 1]^W$  with dimension  $|S|$ . Set  $Y = \bigcup_{|S| \leq m} Y_S$ . Since the union of finitely many closed subsets of dimension at most  $m$  has dimension at most  $m$  [13, page 30 and Theorem V.8],  $Y$  has dimension at most  $m$ .

As  $Z$  is compact, by the assumption on  $\xi$  we can find  $\delta > 0$  such that  $0 < \xi_w - \delta < \xi_w + \delta < 1$  for every  $w \in W$  and the set of  $w \in W$  satisfying  $\xi_w - \delta \leq z_w \leq \xi_w + \delta$  has cardinality at most  $m$  for every  $z \in Z$ . For each  $w \in W$ , take a continuous map  $f_w : [0, 1] \rightarrow [0, 1]$  sending  $[0, \xi_w - \delta]$  to 0, and  $[\xi_w + \delta, 1]$  to 1.

Define a continuous map  $\Psi$  from  $[0, 1]^W$  into itself by setting the coordinate of  $\Psi(x)$  at  $w \in W$  to be  $f_w(x_w)$ . For any  $x \in [0, 1]^W$  and  $w \in W$ , if  $x_w = 0$  or  $1$ , then clearly  $\Psi(x)_w = 0$  or  $1$  accordingly. From the choice of  $\delta$  and  $f_w$  it is also clear that  $\Psi(Z) \subseteq Y$ . Thus  $\dim \Psi(Z) \leq \dim Y \leq m$ .  $\square$

Let  $\theta > 0$ . Take  $F, \delta, M, i$  and  $\xi$  as in Lemma 6.3. By Lemmas 6.5 and 6.4 we can find a continuous map  $\Psi : \Phi_{d_i}(\text{Map}(\rho, F, \delta, \sigma)) \rightarrow [0, 1]^{[d_i] \times \mathcal{U}}$  such that  $\Psi \circ \Phi_{d_i}$  is  $\mathcal{U}^{d_i}|_{\text{Map}(\rho, F, \delta, \sigma)}$ -compatible and  $\dim \Psi(\Phi_{d_i}(\text{Map}(\rho, F, \delta, \sigma))) \leq (\text{mdim}_{\Sigma, M}(X, \rho) + \theta)d_i$ . From Lemma 3.4 we get  $\mathcal{D}(\mathcal{U}, \rho, F, \delta, \delta_i) \leq (\text{mdim}_{\Sigma, M}(X, \rho) + \theta)d_i$ . It follows that

$$\mathcal{D}_{\Sigma}(\mathcal{U}) = \mathcal{D}_{\Sigma}(\mathcal{U}, \rho) \leq \mathcal{D}_{\Sigma}(\mathcal{U}, \rho, F, \delta) \leq \text{mdim}_{\Sigma, M}(X, \rho) + \theta.$$

Since  $\mathcal{U}$  is an arbitrary finite open cover of  $X$  and  $\theta$  is an arbitrary positive number, we get  $\text{mdim}_{\Sigma}(X) \leq \text{mdim}_{\Sigma, M}(X, \rho)$ . As  $\rho$  is an arbitrary compatible metric on  $X$ , we get  $\text{mdim}_{\Sigma}(X) \leq \text{mdim}_{\Sigma, M}(X)$ . This finishes the proof of Theorem 6.1.

The Pontrjagin-Shnirelmann theorem [24] [21, page 80] says that for any compact metrizable space  $Z$ , the dimension  $\dim Z$  of  $Z$  is equal to the minimal value of  $\underline{\dim}_B(Z, \rho)$  for  $\rho$  ranging over compatible metrics on  $Z$ . Since  $\text{mdim}_{\Sigma}(X)$  and  $\text{mdim}_{\Sigma, M}(X, \rho)$  are dynamical analogues of  $\dim(X)$  and  $\underline{\dim}_B(X, \rho)$  respectively, it is natural to ask



**Question 6.6.** Let a countably infinite sofic group  $G$  act continuously on a compact metrizable space  $X$ . Then is there any compatible metric  $\rho$  on  $X$  satisfying

$$\text{mdim}_\Sigma(X) = \text{mdim}_{\Sigma, M}(X, \rho)?$$

Question 6.6 was answered affirmatively by Lindenstrauss in the case  $G = \mathbb{Z}$  and  $X$  has a nontrivial minimal factor [19, Theorem 4.3].

## 7. BERNOULLI SHIFTS

In this section we discuss the sofic mean dimension of the Bernoulli shifts and their factors, proving Theorems 7.1 and 7.4. Throughout this section  $G$  will be a countable sofic group, and  $\Sigma$  will be a fixed sofic approximation sequence of  $G$ .

For any compact metrizable space  $Z$ , we have the left shift action  $\alpha$  of  $G$  on  $Z^G$  given by  $(sx)_t = x_{s^{-1}t}$  for all  $x \in Z^G$  and  $s, t \in G$ .

**Theorem 7.1.** *Let  $Z$  be a compact metrizable space, and consider the left shift action of a countable sofic group  $G$  on  $X = Z^G$ . Then*

$$\text{mdim}_\Sigma(X) \leq \text{mdim}_{\Sigma, M}(X) \leq \dim Z.$$

*If furthermore  $Z$  contains a copy of  $[0, 1]^n$  for every natural number  $n \leq \dim Z$  (for example,  $Z$  could be any polyhedron or the Hilbert cube), then*

$$\text{mdim}_\Sigma(X) = \text{mdim}_{\Sigma, M}(X) = \dim Z.$$

The amenable group case of Theorem 7.1 was proved by Lindenstrauss and Weiss [20, Propositions 3.1 and 3.3], and Coornaert and Krieger [6, Corollaries 4.2 and 5.5].

Recall the lower box dimension recalled at the beginning of Section 4. We need the following lemma.

**Lemma 7.2.** *Let  $Z$  be a compact metrizable space, and consider the left shift action  $\alpha$  of  $G$  on  $X = Z^G$ . Let  $\rho$  be a compatible metric on  $Z$ . Define  $\rho'$  by  $\rho'(x, y) = \rho(x_{e_G}, y_{e_G})$  for  $x, y \in X$ . Then  $\rho'$  is a dynamically generating continuous pseudometric on  $X$ . Furthermore, for any  $\varepsilon > 0$  one has*

$$N_\varepsilon(Z, \rho) \leq h_{\Sigma, \infty}^\varepsilon(\rho') \leq N_{\varepsilon/2}(Z, \rho).$$

*In particular,  $\text{mdim}_{\Sigma, M}(X, \rho') = \underline{\dim}_B(Z, \rho)$ .*

*Proof.* Clearly  $\rho'$  is a dynamically generating continuous pseudometric on  $X$ . Let  $\varepsilon > 0$ .

We show first  $h_{\Sigma, \infty}^\varepsilon(\rho') \geq N_\varepsilon(Z, \rho)$ . It suffices to show  $h_{\Sigma, \infty}^\varepsilon(\rho', F, \delta) \geq N_\varepsilon(Z, \rho)$  for every nonempty finite subset  $F$  of  $G$  and every  $\delta > 0$ . Take a  $(\rho, \varepsilon)$ -separated subset  $Y$  of  $Z$  with  $|Y| = N_\varepsilon(Z, \rho)$ .

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  from some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  such that  $\sqrt{1 - |\Omega|/d} \cdot \text{diam}(Z, \rho) \leq \delta$ , where

$$\Omega = \{a \in [d] : \sigma_{e_G} \circ \sigma_s(a) = \sigma_s(a) \text{ for all } s \in F\}.$$

For each map  $f : [d] \rightarrow Y$ , we define a map  $\varphi_f : [d] \rightarrow X$  by

$$(\varphi_f(a))_t = f(\sigma_{t^{-1}}(a))$$

for all  $a \in [d]$  and  $t \in G$ . Let  $s \in F$ . For any  $a \in \mathcal{Q}$ , one has

$$(\alpha_s \circ \varphi_f(a))_{e_G} = (\varphi_f(a))_{s^{-1}} = f(\sigma_s(a)) = f(\sigma_{e_G} \circ \sigma_s(a)) = (\varphi_f \circ \sigma_s(a))_{e_G},$$

and hence  $\rho'(\alpha_s \circ \varphi_f(a), \varphi_f \circ \sigma_s(a)) = 0$ . Thus

$$\rho'_2(\alpha_s \circ \varphi_f, \varphi_f \circ \sigma_s) \leq \sqrt{(1 - |\mathcal{Q}|/d)(\text{diam}(Z, \rho))^2} \leq \delta.$$

Therefore  $\varphi_f \in \text{Map}(\rho', F, \delta, \sigma)$ . For any distinct maps  $f, g : [d] \rightarrow Y$ , say  $f(a) \neq g(a)$  for some  $a \in [d]$ , one has

$$\begin{aligned} \rho'_\infty(\varphi_f, \varphi_g) &\geq \rho'(\varphi_f(\sigma_{e_G}^{-1}(a)), \varphi_g(\sigma_{e_G}^{-1}(a))) \\ &= \rho((\varphi_f(\sigma_{e_G}^{-1}(a)))_{e_G}, (\varphi_g(\sigma_{e_G}^{-1}(a)))_{e_G}) \\ &= \rho(f(a), g(a)) \geq \varepsilon. \end{aligned}$$

Thus the set  $\{\varphi_f : f \in Y^{[d]}\}$  is  $(\rho'_\infty, \varepsilon)$ -separated. Therefore

$$N_\varepsilon(\text{Map}(\rho', F, \delta, \sigma), \rho'_\infty) \geq |Y|^d = (N_\varepsilon(Z, \rho))^d.$$

It follows that  $h_{\Sigma, \infty}^\varepsilon(\rho', F, \delta) \geq N_\varepsilon(Z, \rho)$  as desired.

Next we show  $h_{\Sigma, \infty}^\varepsilon(\rho') \leq N_{\varepsilon/2}(Z, \rho)$ . It suffices to show  $h_{\Sigma, \infty}^\varepsilon(\rho', \{e_G\}, 1) \leq N_{\varepsilon/2}(Z, \rho)$ . Take a maximal  $(\rho, \varepsilon/2)$ -separated subset  $Y$  of  $Z$ . Then  $|Y| \leq N_{\varepsilon/2}(Z, \rho)$ .

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  from some  $d \in \mathbb{N}$ . Let  $\mathcal{E}$  be a  $(\rho'_\infty, \varepsilon)$ -separated subset of  $\text{Map}(\rho', \{e_G\}, 1, \sigma)$  with  $|\mathcal{E}| = N_\varepsilon(\text{Map}(\rho', \{e_G\}, 1, \sigma), \rho'_\infty)$ . For each  $\varphi \in \mathcal{E}$ , we find some map  $f_\varphi : [d] \rightarrow Y$  such that

$$\max_{a \in [d]} \rho((\varphi(a))_{e_G}, f_\varphi(a)) < \varepsilon/2.$$

If  $\varphi, \psi \in \mathcal{E}$  and  $f_\varphi = f_\psi$ , then

$$\rho'_\infty(\varphi, \psi) = \max_{a \in [d]} \rho((\varphi(a))_{e_G}, (\psi(a))_{e_G}) < \varepsilon,$$

and hence  $\varphi = \psi$ , since  $\mathcal{E}$  is  $(\rho'_\infty, \varepsilon)$ -separated. Therefore

$$N_\varepsilon(\text{Map}(\rho', \{e_G\}, 1, \sigma), \rho'_\infty) = |\mathcal{E}| \leq |Y|^d \leq (N_{\varepsilon/2}(Z, \rho))^d.$$

It follows that  $h_{\Sigma, \infty}^\varepsilon(\rho', \{e_G\}, 1) \leq N_{\varepsilon/2}(Z, \rho)$  as desired.  $\square$

We also need the following version of Lebesgue's covering theorem:

**Lemma 7.3.** *[20, Lemma 3.2] [13, Theorem IV.2] Let  $W$  be a finite set. Let  $\mathcal{U}$  be a finite open cover of  $[0, 1]^W$  such that no item of  $\mathcal{U}$  intersects two opposing sides of  $[0, 1]^W$ . Then*

$$\mathcal{D}(\mathcal{U}) \geq |W|.$$

We are ready to prove Theorem 7.1.

*Proof of Theorem 7.1.* By Theorem 6.1, Lemma 7.2, Proposition 4.4 and the Pontrjagin-Shnirelmann theorem as recalled at the end of Section 6, we have  $\text{mdim}_\Sigma(X) \leq \text{mdim}_{\Sigma, M}(X) \leq \dim Z$ .

Now assume that  $Z$  contains a copy of  $[0, 1]^n$  for every natural number  $n \leq \dim Z$ . It suffices to show  $\text{mdim}_\Sigma(X) \geq \dim Z$ . In turn it suffices to show  $\text{mdim}_\Sigma(X) \geq n$  for every natural number  $n \leq \dim Z$ . Since  $([0, 1]^n)^G$  is a closed  $G$ -invariant subset of  $Z^G$ , by Proposition 4.5 one has  $\text{mdim}_\Sigma(X) \geq \text{mdim}_\Sigma(([0, 1]^n)^G)$ . Therefore it suffices to show  $\text{mdim}_\Sigma(([0, 1]^n)^G) \geq n$ .

Take a finite open cover  $\mathcal{U}$  of  $[0, 1]^n$  such that no item of  $\mathcal{U}$  intersects two opposing sides of  $[0, 1]^n$ . For each  $d \in \mathbb{N}$ , note that no item of  $\mathcal{U}^d$  intersects two opposing sides of  $([0, 1]^n)^{[d]} = [0, 1]^{dn}$ , and hence by Lemma 7.3 one has  $\mathcal{D}(\mathcal{U}^d) \geq dn$ .

Denote by  $\pi$  the map  $([0, 1]^n)^G \rightarrow [0, 1]^n$  sending  $x$  to  $x_{e_G}$ . Then  $\tilde{\mathcal{U}} := \pi^{-1}(\mathcal{U})$  is a finite open cover of  $([0, 1]^n)^G$ . Let  $\rho$  be a compatible metric on  $([0, 1]^n)^G$ . It suffices to show  $\mathcal{D}_\Sigma(\tilde{\mathcal{U}}, \rho, F, \delta) \geq n$  for every nonempty finite subset  $F$  of  $G$  and every  $\delta > 0$ .

Take a nonempty finite subset  $K$  of  $G$  such that if  $x, y \in ([0, 1]^n)^G$  are equal on  $K$ , then  $\rho(x, y) < \delta/2$ .

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  such that  $\delta^2/4 + (1 - |\mathcal{Q}|/d)(|F| + 1)(\text{diam}(([0, 1]^n)^G, \rho))^2 \leq \delta^2$ , where

$$\mathcal{Q} = \{a \in [d] : \sigma_{t^{-1}} \circ \sigma_s(a) = \sigma_{t^{-1}s}(a) \text{ for all } s \in F, t \in K \text{ and } \sigma_{e_G}(a) = a\}.$$

For each map  $f : [d] \rightarrow [0, 1]^n$ , we define a map  $\varphi_f : [d] \rightarrow ([0, 1]^n)^G$  by

$$(2) \quad (\varphi_f(a))_t = f(\sigma_{t^{-1}}(a))$$

for all  $a \in [d]$  and  $t \in G \setminus \{e_G\}$ , and

$$(\varphi_f(a))_{e_G} = f(a)$$

for all  $a \in [d]$ . Note that (2) holds for all  $a \in \mathcal{Q}$  and  $t \in G$ . Set  $\mathcal{R} = \mathcal{Q} \cap \bigcap_{s \in F} \sigma_s^{-1}(\mathcal{Q})$ . For any  $a \in \mathcal{R}$ ,  $s \in F$ , and  $t \in K$ , since  $a, \sigma_s(a) \in \mathcal{Q}$ , one has

$$(\alpha_s \circ \varphi_f(a))_t = (\varphi_f(a))_{s^{-1}t} = f(\sigma_{t^{-1}s}(a)) = f(\sigma_{t^{-1}} \circ \sigma_s(a)) = (\varphi_f \circ \sigma_s(a))_t,$$

and hence  $\rho(\alpha_s \circ \varphi_f(a), \varphi_f \circ \sigma_s(a)) < \delta/2$  by the choice of  $K$ . Thus

$$\begin{aligned} \rho_2(\alpha_s \circ \varphi_f, \varphi_f \circ \sigma_s) &\leq \sqrt{\delta^2/4 + (1 - |\mathcal{R}|/d)(\text{diam}(([0, 1]^n)^G, \rho))^2} \\ &\leq \sqrt{\delta^2/4 + (1 - |\mathcal{Q}|/d)(|F| + 1)(\text{diam}(([0, 1]^n)^G, \rho))^2} \leq \delta. \end{aligned}$$

Therefore  $\varphi_f \in \text{Map}(\rho, F, \delta, \sigma)$ . The map  $\Phi : ([0, 1]^n)^{[d]} \rightarrow \text{Map}(\rho, F, \delta, \sigma)$  sending  $f$  to  $\varphi_f$  is clearly continuous. Note that  $\Phi^{-1}(\tilde{\mathcal{U}}^d|_{\text{Map}(\rho, F, \delta, \sigma)}) = \mathcal{U}^d$ . Therefore  $\mathcal{D}_\Sigma(\tilde{\mathcal{U}}, \rho, F, \delta, \sigma) \geq \mathcal{D}(\mathcal{U}^d) \geq dn$ . It follows that  $\mathcal{D}_\Sigma(\tilde{\mathcal{U}}, \rho, F, \delta) \geq n$  as desired.  $\square$

**Theorem 7.4.** *Let  $Z$  be a path-connected compact metrizable space, and consider the left shift action  $\alpha$  of a countable sofic group  $G$  on  $X = Z^G$ . For any nontrivial*

factor  $Y$  of  $X$ , one has

$$\text{mdim}_\Sigma(Y) > 0.$$

The amenable group case of Theorem 7.4 was proved by Lindenstrauss and Weiss [20, Theorem 3.6]. We adapt their argument to our situation.

**Lemma 7.5.** *Let  $G$  act continuously on a compact metrizable space  $X$ . Let  $Y$  be a factor of  $X$  with factor map  $\pi : X \rightarrow Y$ . Let  $\mathcal{U}$  be a finite open cover of  $Y$ . Then*

$$\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U})) \leq \mathcal{D}_\Sigma(\mathcal{U}).$$

*Proof.* Let  $\rho$  and  $\rho'$  be compatible metrics on  $X$  and  $Y$  respectively. Replacing  $\rho$  by  $\rho + \pi^{-1}(\rho')$  if necessary, we may assume that  $\rho(x, y) \geq \rho'(\pi(x), \pi(y))$  for all  $x, y \in X$ .

Let  $F$  be a nonempty finite subset of  $X$  and  $\delta > 0$ . Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ . For any  $\varphi \in \text{Map}(\rho, F, \delta, \sigma)$ , one has  $\pi \circ \sigma \in \text{Map}(\rho', F, \delta, \sigma)$ . Thus we have a continuous map  $\Phi : \text{Map}(\rho, F, \delta, \sigma) \rightarrow \text{Map}(\rho', F, \delta, \sigma)$  sending  $\varphi$  to  $\pi \circ \varphi$ . Furthermore,  $\Phi^{-1}(\mathcal{U}^d|_{\text{Map}(\rho', F, \delta, \sigma)}) = (\pi^{-1}(\mathcal{U}))^d|_{\text{Map}(\rho, F, \delta, \sigma)}$ . Thus  $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho, F, \delta, \sigma) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho', F, \delta, \sigma)$ . It follows that  $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho, F, \delta) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho', F, \delta)$ . Since  $F$  is an arbitrary nonempty finite subset of  $G$  and  $\delta$  is an arbitrary positive number, we get

$$\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U})) = \mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho) \leq \mathcal{D}_\Sigma(\mathcal{U}, \rho') = \mathcal{D}_\Sigma(\mathcal{U}).$$

□

We are ready to prove Theorem 7.4.

*Proof of Theorem 7.4.* Denote by  $\pi$  the factor map  $X \rightarrow Y$ . Let  $\mathcal{U} = \{U, V\}$  be an open cover of  $Y$  such that none of  $U$  and  $V$  is dense in  $Y$ . By Lemma 7.5 it suffices to show that  $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U})) > 0$ . Take compatible metrics  $\rho$  and  $\rho'$  on  $Z$  and  $X$  respectively.

Note that none of  $\pi^{-1}(U)$  and  $\pi^{-1}(V)$  is dense in  $X$ . Take  $x_U \in X \setminus \overline{\pi^{-1}(U)}$  and  $x_V \in X \setminus \overline{\pi^{-1}(V)}$ . Then there exist a finite symmetric subset  $K$  of  $G$  containing  $e_G$  such that if  $x \in X$  coincides with  $x_U$  (resp.  $x_V$ ) on  $K$ , then  $x \notin \pi^{-1}(U)$  (resp.  $x \notin \pi^{-1}(V)$ ). Since  $Z$  is path connected, for each  $s \in K$  we can take a continuous map  $\gamma_s : [0, 1] \rightarrow Z$  such that  $\gamma_s(0) = (x_U)_s$  and  $\gamma_s(1) = (x_V)_s$ .

Now it suffices to show  $\mathcal{D}_\Sigma(\pi^{-1}(\mathcal{U}), \rho', F, \delta) \geq 1/(2|K^2|)$  for every finite subset  $F$  of  $G$  containing  $K$  and every  $\delta > 0$ . Take a finite symmetric subset  $K_1$  of  $G$  such that if two points  $x$  and  $y$  of  $X$  coincide on  $K_1$ , then  $\rho'(x, y) < \delta/2$ .

Let  $\sigma$  be a map from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$  which is a good enough sofic approximation for  $G$  such that  $\sqrt{\delta^2/4 + (1 - |\mathcal{Q}|/d)(\text{diam}(X, \rho'))^2} \leq \delta$  and  $|\mathcal{Q}|/d \geq 1/2$ , where

$$\begin{aligned} \mathcal{Q} = \{a \in [d] : \sigma_s \sigma_t(a) = \sigma_{st}(a) \text{ for all } s, t \in F \cup K_1, \text{ and } \sigma_{e_G}(a) = a, \\ \text{and } \sigma_s(a) \neq \sigma_t(a) \text{ for all distinct } s, t \in K\}. \end{aligned}$$

Take a maximal subset  $\mathcal{W}$  of  $\mathcal{Q}$  subject to the condition that the sets  $\sigma(K)a$  for  $a \in \mathcal{W}$  are pairwise disjoint. Then  $\mathcal{Q} \subseteq \sigma(K^2)\mathcal{W}$ . Thus  $|\mathcal{W}|/d \geq |\mathcal{Q}|/(|K^2|d) \geq 1/(2|K^2|)$ .

Now we define a map  $\Phi : [0, 1]^{\mathcal{W}} \rightarrow \text{Map}(\rho', F, \delta, \sigma)$ . Fix  $z_0 \in Z$ . Let  $f \in [0, 1]^{\mathcal{W}}$ . We define  $\tilde{f} \in Z^{[d]}$  by

$$\tilde{f}_{\sigma(s)a} = \gamma_{s^{-1}}(f(a))$$

for  $a \in \mathcal{W}$ ,  $s \in K$ , and

$$\tilde{f}_b = z_0$$

for  $b \in [d] \setminus \sigma(K)\mathcal{W}$ . Then we define  $\varphi_{\tilde{f}} \in X^{[d]}$  by

$$(\varphi_{\tilde{f}}(a))_s = \tilde{f}(\sigma_{s^{-1}}(a))$$

for  $a \in [d]$  and  $s \in G$ . Let  $s \in F$ . For any  $a \in \mathcal{Q}$  and  $t \in K_1$ , one has

$$(\alpha_s \circ \varphi_{\tilde{f}}(a))_t = (\varphi_{\tilde{f}}(a))_{s^{-1}t} = \tilde{f}(\sigma_{t^{-1}s}(a)) = \tilde{f}(\sigma_{t^{-1}} \circ \sigma_s(a)) = (\varphi_{\tilde{f}} \circ \sigma_s(a))_t,$$

and hence  $\rho'(\alpha_s \circ \varphi_{\tilde{f}}(a), \varphi_{\tilde{f}} \circ \sigma_s(a)) < \delta/2$  by the choice of  $K_1$ . Thus

$$\rho'_2(\alpha_s \circ \varphi_{\tilde{f}}, \varphi_{\tilde{f}} \circ \sigma_s) \leq \sqrt{\delta^2/4 + (1 - |\mathcal{Q}|/d)(\text{diam}(X, \rho'))^2} \leq \delta.$$

Therefore  $\varphi_{\tilde{f}} \in \text{Map}(\rho', F, \delta, \sigma)$ . Set  $\Phi(f) = \varphi_{\tilde{f}}$ . Clearly  $\Phi$  is continuous.

Set  $\mathcal{V} = \Phi^{-1}((\pi^{-1}(\mathcal{U}))^{[d]}|_{\text{Map}(\rho', F, \delta, \sigma)})$ . Let  $f \in [0, 1]^{\mathcal{W}}$  and  $a \in \mathcal{W}$ . If  $f(a) = 0$ , then  $(\varphi_{\tilde{f}}(a))_s = \tilde{f}_{\sigma(s^{-1})a} = \gamma_s(0) = (x_U)_s$  for all  $s \in K$ , and hence  $\varphi_{\tilde{f}}(a) \notin \pi^{-1}(U)$  by the choice of  $K$ . Similarly, if  $f(a) = 1$ , then  $\varphi_{\tilde{f}}(a) \notin \pi^{-1}(V)$ . Thus no item of  $\mathcal{V}$  intersects two opposing sides of  $[0, 1]^{\mathcal{W}}$ . By Lemma 7.3 we conclude that  $\mathcal{D}(\pi^{-1}(\mathcal{U}), \rho', F, \delta, \sigma) \geq \mathcal{D}(\mathcal{V}) \geq |\mathcal{W}| \geq d/(2|K^2|)$ . It follows that  $\mathcal{D}_{\Sigma}(\pi^{-1}(\mathcal{U}), \rho', F, \delta) \geq q/(2|K^2|)$  as desired.  $\square$

## 8. SMALL-BOUNDARY PROPERTY

In this section we discuss the relation between the small-boundary property and non-positive sofic mean topological dimension.

We start with recalling the definitions of zero inductive dimensional compact metrizable spaces and actions with small-boundary property.

A compact metrizable space  $Y$  is said to have *inductive dimension* 0 if for every  $y \in Y$  and every neighborhood  $U$  of  $Y$  there exists a neighborhood  $V$  of  $y$  contained in  $U$  such that the boundary  $\partial V$  of  $V$  is empty [13, Definition II.1]. A compact metrizable space has inductive dimension 0 if and only if it has covering dimension 0 [13, Theorem V.8].

**Definition 8.1.** Let a countable group  $\Gamma$  act continuously on a compact metrizable space  $X$ . We denote by  $M(X, \Gamma)$  the set of  $\Gamma$ -invariant Borel probability measures on  $X$ . We say that a closed subset  $Z$  of  $X$  is *small* if  $\mu(Z) = 0$  for all  $\mu \in M(X, \Gamma)$ . In particular, when  $M(X, \Gamma)$  is empty, every closed subset of  $X$  is small. We say that the action has the *small-boundary property* (SBP) if for every point  $x \in X$  and every neighborhood  $U$  of  $x$ , there is a neighborhood  $V \subseteq U$  of  $x$  with small boundary.

When  $\Gamma$  is amenable, for any subset  $Z$  of  $X$  it is easy to check that the function  $F \mapsto \max_{x \in X} \sum_{s \in F} 1_Z(sx)$  defined on the set of nonempty finite subsets of  $\Gamma$  satisfies the conditions of the Ornstein-Weiss lemma [20, Theorem 6.1]. Thus  $\frac{1}{|F|} \max_{x \in X} \sum_{s \in F} 1_Z(sx)$  converges to some limit as  $F$  becomes more and more left invariant. Shub and Weiss defined  $Z$  to be *small* if this limit is 0 [25]. It is proved in page 538 of [25] that when  $\Gamma$  is amenable and  $Z$  is closed, the definition of Shub and Weiss coincides with ours. The notion of the SBP was introduced in [19] and [20].

If  $X$  has less than  $2^{\aleph_0}$  ergodic  $G$ -invariant Borel probability measures, then the action has the SBP [25] [20, page 18].

When  $\Gamma$  is amenable, Lindenstrauss and Weiss showed that actions with the SBP has zero mean topological dimension [20, Theorem 5.4]. We extend their result to sofic case:

**Theorem 8.2.** *Let a countable sofic group  $G$  act continuously on a compact metrizable space  $X$ . Suppose that the action has the SBP. Let  $\Sigma$  be a sofic approximation sequence for  $G$ . Then  $\text{mdim}_\Sigma(X) \leq 0$ .*

Lindenstrauss showed that if a continuous action of  $\mathbb{Z}$  on a compact metrizable space has a nontrivial minimal factor and has zero mean topological dimension, then it has the SBP [19, Theorem 6.2]. Gutman showed that if a continuous action of  $\mathbb{Z}^d$  for  $d \in \mathbb{N}$  on a compact metrizable space has a free zero-dimensional factor and has zero mean topological dimension, then it has the SBP [12, Theorem 1.11.1]. It is not clear how generally the converse of Theorem 8.2 could be true (see the discussion on page 20 of [20]).

We need the following three lemmas.

**Lemma 8.3.** *Let a countable group  $\Gamma$  act continuously on a compact metrizable space  $X$ . Let  $Z$  be a closed subset of  $X$ . Then for any  $\varepsilon > 0$  there is an open neighborhood  $U$  of  $Z$  with  $\sup_{\mu \in M(X, \Gamma)} \mu(U) < \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$ .*

*Proof.* Suppose that this not true. Then  $M(X, \Gamma)$  is nonempty, and for each open neighborhood  $U$  of  $Z$  there is some  $\mu_U \in M(X, \Gamma)$  with  $\mu_U(U) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$ . The set of open neighborhoods of  $Z$  is partially ordered by reversing inclusion. Take a limit point  $\nu$  of this net  $\{\mu_U\}_U$  in the compact space  $M(X, \Gamma)$ . We claim that  $\nu(Z) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$ , which is a contradiction.

To prove the claim, by the regularity of  $\nu$ , it suffices to show  $\nu(V) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$  for every open neighborhood  $V$  of  $Z$ . Take an open neighborhood  $U'$  of  $Z$  such that  $\overline{U'} \subseteq V$ . Take a continuous function  $f : X \rightarrow [0, 1]$  such that  $f = 1$  on  $\overline{U'}$  and  $f = 0$  on  $X \setminus V$ . For any open neighborhood  $U$  of  $Z$  satisfying  $U \subseteq U'$ , one has  $\int_X f d\mu_U \geq \mu_U(U) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$ . It follows that  $\nu(V) \geq \int_X f d\nu \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$ , as desired.  $\square$

**Lemma 8.4.** *Consider a continuous action of a countable group  $\Gamma$  on a compact metrizable space  $X$  with the SBP. Then for any finite open cover  $\mathcal{U}$  of  $X$  and any  $\varepsilon > 0$  there is a partition of unity  $\phi_j : X \rightarrow [0, 1]$  for  $j = 1, \dots, |\mathcal{U}|$  subordinate to  $\mathcal{U}$  such that  $\sup_{\mu \in M(X, \Gamma)} \mu(\bigcup_{j=1}^{|\mathcal{U}|} \phi_j^{-1}(0, 1)) < \varepsilon$ .*

*Proof.* For each  $x \in X$ , take a neighborhood  $V_x$  of  $x$  with small boundary such that  $\overline{V_x}$  is contained in some item of  $\mathcal{U}$ . Replacing  $V_x$  by its interior if necessary, we may assume that  $V_x$  is open. Since  $X$  is compact, we can cover  $X$  by finitely many such  $V_x$ 's. Note that if two subsets  $Y_1$  and  $Y_2$  of  $X$  have small boundary  $\partial Y_1$  and  $\partial Y_2$  respectively, then  $\partial(Y_1 \cup Y_2) \subseteq \partial Y_1 \cup \partial Y_2$  is also small. Thus we may take the union of those chosen  $V_x$ 's whose closures are contained in one item of  $\mathcal{U}$  to obtain an open cover  $\mathcal{U}'$  of  $X$  such that each item of  $\mathcal{U}'$  has small boundary and there is a bijection  $\varphi : \mathcal{U}' \rightarrow \mathcal{U}$  with  $\overline{U'} \subseteq \varphi(U')$  for every  $U' \in \mathcal{U}'$ .

By Lemma 8.3, for each  $U' \in \mathcal{U}'$  we can find an open neighborhood  $U''$  of the boundary  $\partial U'$  of  $U'$  such that  $\sup_{\mu \in M(X, \Gamma)} \mu(U'') < \varepsilon/|\mathcal{U}|$ . Replacing  $U''$  by  $U'' \cap \varphi(U')$  if necessary, we may assume that  $U'' \subseteq \varphi(U')$ . Take an open neighborhood  $U'''$  of  $\partial U'$  such that  $\overline{U'''} \subseteq U''$ . List the items of  $\mathcal{U}'$  as  $U'_1, \dots, U'_{|\mathcal{U}|}$ . For each  $1 \leq j \leq |\mathcal{U}|$ , take a continuous function  $\psi_j : X \rightarrow [0, 1]$  such that  $\psi_j = 1$  on  $\overline{U'_j}$  and  $\psi_j = 0$  on  $X \setminus (U'_j \cup U'''_j)$ . Now define  $\phi_j$  for  $1 \leq j \leq |\mathcal{U}|$  inductively as  $\phi_1 = \psi_1$ , and  $\phi_j = \min(\psi_j, 1 - \sum_{i=1}^{j-1} \phi_i)$  for  $2 \leq j \leq |\mathcal{U}|$ . Then  $\phi_1, \dots, \phi_{|\mathcal{U}|}$  is a partition of unity subordinate to  $\mathcal{U}$ . Furthermore, one has

$$\bigcup_{j=1}^{|\mathcal{U}|} \phi_j^{-1}(0, 1) \subseteq \bigcup_{j=1}^{|\mathcal{U}|} \psi_j^{-1}(0, 1) \subseteq \bigcup_{j=1}^{|\mathcal{U}|} U'''_j \subseteq \bigcup_{j=1}^{|\mathcal{U}|} U''_j,$$

and hence  $\mu(\bigcup_{j=1}^{|\mathcal{U}|} \phi_j^{-1}(0, 1)) \leq \mu(\bigcup_{j=1}^{|\mathcal{U}|} U''_j) < \varepsilon$  for every  $\mu \in M(X, \Gamma)$ .  $\square$

**Lemma 8.5.** *Let a countable group  $\Gamma$  act continuously on a compact metrizable space  $X$ . Let  $\rho$  be a compatible metric on  $X$ . Let  $Z$  be a closed subset of  $X$ , and  $\varepsilon > 0$ . Then there exist a nonempty finite subset  $F$  of  $\Gamma$  and  $\delta > 0$  such that, for any map  $\sigma$  from  $\Gamma$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ , one has*

$$\frac{1}{d} \max_{\varphi \in \text{Map}(\rho, F, \delta, \sigma)} \sum_{a \in [d]} 1_Z(\varphi(a)) < \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z).$$

*Proof.* Suppose that for any nonempty finite subset  $F$  of  $\Gamma$  and any  $\delta > 0$ , there are a map  $\sigma_{F, \delta}$  from  $\Gamma$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$  and a  $\varphi_{F, \delta} \in \text{Map}(\rho, F, \delta, \sigma_{F, \delta})$  with

$$\frac{1}{d} \sum_{a \in [d]} 1_Z(\varphi_{F, \delta}(a)) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z).$$

Denote by  $\mu_{F, \delta}$  the probability measure  $\frac{1}{d} \sum_{a \in [d]} \delta_{\varphi_{F, \delta}(a)}$  on  $X$ . The set of all such  $(F, \delta)$  is partially ordered by  $(F, \delta) \geq (F', \delta')$  when  $F \supseteq F', \delta \leq \delta'$ . Take a limit point  $\nu$  of  $\{\mu_{F, \delta}\}_{F, \delta}$  in the compact set of Borel probability measures on  $X$ .

We claim that  $\nu$  is  $G$ -invariant. Let  $g$  be a continuous  $\mathbb{R}$ -valued function on  $X$ . For each  $\delta > 0$ , set  $C_\delta = \max_{x,y \in X, \rho(x,y) \leq \delta} |g(x) - g(y)|$ . For any  $(F, \delta)$  as above and any  $s \in F$ , since  $\varphi_{F,\delta} \in \text{Map}(\rho, F, \delta, \sigma_{F,\delta})$ , we have  $|\mathcal{W}_s| \geq d(1 - \delta)$ , where

$$\mathcal{W}_s = \{a \in [d] : \rho(\varphi_{F,\delta}(sa), s\varphi_{F,\delta}(a)) \leq \sqrt{\delta}\}.$$

Thus, for any  $s \in F$ , one has

$$\begin{aligned} |\mu_{F,\delta}(g) - (s\mu_{F,\delta})(g)| &= \frac{1}{d} \left| \sum_{a \in [d]} g(\varphi_{F,\delta}(a)) - \sum_{a \in [d]} g(s\varphi_{F,\delta}(a)) \right| \\ &= \frac{1}{d} \left| \sum_{a \in [d]} g(\varphi_{F,\delta}(sa)) - \sum_{a \in [d]} g(s\varphi_{F,\delta}(a)) \right| \\ &\leq \frac{1}{d} \sum_{a \in [d]} |g(\varphi_{F,\delta}(sa)) - g(s\varphi_{F,\delta}(a))| \\ &\leq \frac{1}{d} (|\mathcal{W}_s| C_{\sqrt{\delta}} + (d - |\mathcal{W}_s|) 2\|g\|_\infty) \\ &\leq C_{\sqrt{\delta}} + 2\delta\|g\|_\infty. \end{aligned}$$

Taking a net of  $(F, \delta) \rightarrow \infty$  with  $\mu_{F,\delta} \rightarrow \nu$ , we have  $\mu_{F,\delta}(g) \rightarrow \nu(g)$  and  $(s\mu_{F,\delta})(g) \rightarrow (s\nu)(g)$  for every  $s \in G$ , and  $\delta, C_{\sqrt{\delta}} \rightarrow 0$ . It follows that  $\nu(g) = (s\nu)(g)$ . This proves the claim.

Next we claim that  $\nu(Z) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$ , which is a contradiction.

To prove the claim, by the regularity of  $\nu$ , it suffices to show  $\nu(U) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$  for every open neighborhood  $U$  of  $Z$ . Take a continuous function  $f : X \rightarrow [0, 1]$  such that  $f = 1$  on  $Z$  and  $f = 0$  on  $X \setminus U$ . For any  $(F, \delta)$  as above one has

$$\mu_{F,\delta}(f) \geq \frac{1}{d} \sum_{a \in [d]} 1_Z(\varphi_{F,\delta}(a)) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z).$$

Letting  $(F, \delta) \rightarrow \infty$  with  $\mu_{F,\delta} \rightarrow \nu$ , we get  $\nu(U) \geq \nu(f) \geq \varepsilon + \sup_{\mu \in M(X, \Gamma)} \mu(Z)$  as desired.  $\square$

We are ready to prove Theorem 8.2.

*Proof of Theorem 8.2.* Fix a compatible metric  $\rho$  on  $X$ . Let  $\mathcal{U}$  be a finite open cover of  $X$ . Set  $k = |\mathcal{U}|$ . Let  $\varepsilon > 0$ . Take  $\phi_1, \dots, \phi_k$  as in Lemma 8.4 for  $\mathcal{U}$  and  $\varepsilon$ . Set  $Z = \bigcup_{j=1}^k \phi_j^{-1}(0, 1)$ . Then  $\sup_{\mu \in M(X, G)} \mu(Z) \leq \varepsilon$ . By Lemma 8.5 we can find a nonempty finite subset  $F$  of  $G$  and  $\delta > 0$  such that, for any map  $\sigma$  from  $G$  to  $\text{Sym}(d)$  for some  $d \in \mathbb{N}$ , one has

$$\frac{1}{d} \max_{\varphi \in \text{Map}(\rho, F, \delta, \sigma)} \sum_{a \in [d]} 1_Z(\varphi(a)) < \varepsilon + \sup_{\mu \in M(X, G)} \mu(Z) \leq 2\varepsilon.$$



Define  $\Phi : X \rightarrow \mathbb{R}^k$  by  $\Phi(x) = (\phi_1(x), \dots, \phi_k(x))$ . Define  $\Phi_{F,\delta,\sigma} : \text{Map}(\rho, F, \delta, \sigma) \rightarrow \mathbb{R}^{kd}$  by

$$\Phi_{F,\delta,\sigma}(\varphi) = (\Phi(\varphi(1)), \dots, \Phi(\varphi(d))).$$

Let  $e_j^i$ ,  $i = 1, \dots, d, j = 1, \dots, k$  be the standard basis of  $\mathbb{R}^{kd}$ . For every  $I \subseteq [d]$  with  $|I| \leq 2\epsilon d$  and every  $\xi \in \{0, 1\}^{kd}$ , define

$$C(I, \xi) = \text{span}\{e_j^i : i \in I, 1 \leq j \leq k\} + \xi.$$

Then

$$\Phi_{F,\delta,\sigma}(\text{Map}(\rho, F, \delta, \sigma)) \subseteq \bigcup_{|I| \leq 2\epsilon d, \xi} C(I, \xi).$$

Note that  $\Phi_{F,\delta,\sigma}$  is  $\mathcal{U}^d|_{\text{Map}(\rho, F, \delta, \sigma)}$ -compatible, and  $\bigcup_{|I| \leq 2\epsilon d, \xi} C(I, \xi)$  is a finite union of at most  $2\epsilon kd$  dimensional affine subspaces of  $\mathbb{R}^{kd}$ . Since the union of finitely many closed subsets of dimension at most  $2\epsilon kd$  has dimension at most  $2\epsilon kd$  [13, page 30 and Theorem V.8], from Lemma 3.4 we get

$$\mathcal{D}(\mathcal{U}, \rho, F, \delta, \sigma) \leq 2\epsilon kd.$$

It follows that  $\mathcal{D}_\Sigma(\mathcal{U}) \leq 0$ , and hence  $\text{mdim}_\Sigma(X) \leq 0$ .  $\square$

We leave the following

**Question 8.6.** Could one strengthen Theorem 8.2 to get  $\text{mdim}_{\Sigma, M}(X) = 0$ ?

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